



SIX LECTURES
ON
THE MEAN-VALUE THEOREM
OF
THE DIFFERENTIAL CALCULUS

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ON
THE MEAN-VALUE THEOREM
OF
THE DIFFERENTIAL CALCULUS

delivered at the Calcutta University

BY

GANESH PRASAD

HARDINGE PROFESSOR OF HIGHER MATHEMATICS, LIFE-PRESIDENT
OF THE BENARES MATHEMATICAL SOCIETY, AND PRESIDENT
OF THE CALCUTTA MATHEMATICAL SOCIETY



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PREFACE

In this book are contained without any material alteration the six public lectures, which were delivered by me during the first four months of 1930 with the two-fold object of (1) giving an up-to-date account of our knowledge of the mean-value theorem and the function θ to which I have ventured to give the name of Rolle's function, and (2) stimulating research relating to Rolle's function by suggesting problems which were at the time engaging my attention and had not been completely solved.

Apart from the first chapter, which is historical and introductory, the book may be roughly divided into two parts, viz., the second and third chapters which deal with the mean-value theorem and its generalizations, and the fourth and fifth chapters and the major part of the sixth chapter in which Rolle's function has been studied. The first Appendix is of interest because of the correspondence between Professor Pompeiu and myself about his remarkable proof of the mean-value theorem. The second Appendix deals with the history of the various forms of the remainder in Taylor's series. The third Appendix contains corrections and additions.

I venture to say that the chief interest of the book is the prominent place given in it to Rolle's function. It is true that the function was considered by Cauchy about 100 years ago and was later studied by American and Cambridge mathematicians without more being discovered about its functional property than that it may be multiple-valued. It is only very recently that well-known mathematicians, for example, Professor Rudolf Rothe and Professor T. Hayashi, took up the study of the function but in ignorance of much of the earlier work. A few of the landmarks in this field of research may be enumerated here. (1) Professor Rothe had never contemplated the possibility of $\theta(h)$ being non-differentiable; I have given functions $\theta(h)$ which are single-valued, finite and continuous and at the same time without differential coefficients at the points of an everywhere dense set. (2) Prof. Hedrick had attempted the study of $\theta(h)$ as a multiple-valued function, but, for want of the separate treatment of the different values corresponding to a given h , his treatment is confused and infructuous. I have introduced the notion of *principal value* of $\theta(h)$ as the greatest of the different values and shown

that, for $f(h)$ as nowhere differentiable, $\theta(h)$ may be also nowhere differentiable, with the possible exception of the points of the first category where $f(h)$ has "cusps."

It is a great pleasure to me to record my obligations to a number of my friends and pupils who have helped me in various ways, during the time the book was in the Press. To Dr. Bibhutibhusan Datta, D.Sc., I am indebted for his advice relating to the parts which are of historical character. Dr. A. N. Singh, D.Sc., Lecturer in the Lucknow University, has gone through nearly the whole of the book in proof and has given me valuable advice relating to certain parts of it. Mr. Hariprasanna Banerjee, M.Sc., Lecturer in Pure Mathematics to the Post-graduate students in the Calcutta University, has also helped me like Dr. Singh with valuable advice. Some of my present researchers, specially Mr. Bholanath Mookerjee, M.A., P.H.S., of Scottish Church College, Mr. Santoshkumar Bhar, M.Sc., and Mr. Rama Dhar Misra, M.A., have also given me help in going through the proof-sheets.

CALCUTTA:

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CHANDER PRASAD



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FIRST LECTURE

Historical and Introductory.

Colleagues, students and other gentlemen!

The six lectures which I propose to deliver on the mean-value theorem, rightly called the fundamental theorem of the Differential Calculus, are intended to make known to you a field of research, which has only in comparatively recent times attracted anything like considerable attention from prominent mathematicians and which is far from being exhausted. I shall feel amply rewarded if I succeed in inspiring some of you with enthusiasm to take up for research even a few of the numerous problems that may be suggested by my lectures.

§ 1.

1. To-day's lecture will be of a historical and introductory character, and I shall begin by giving you an account of the circumstances through which Rolle's theorem, from which the mean-value theorem originated, has come. Michel Rolle (1682-1719), who was from 1695 onwards a paid member of the Academy of Sciences of Paris, was one of the small band of the "old mathematicians who did not content themselves with the institutional proofs of theorems but demanded rigorous logical definitions and proofs."¹ In a small *dissertation* book,² published in 1691, he gives what goes now by the name of Rolle's theorem,³ viz., $f(x)=0$ has at least one

¹ See Felix Klein's *Vorlesung über Differential- und Integralrechnung und Geometrie*, 1st edition, p. 114.

² *Démonstration d'une Méthode pour résoudre les Équations de tous les degrés; suivie de deux autres Méthodes, dont la première donne les racines de plusieurs cas mêmes équation par la Géométrie et la seconde, pour résoudre plusieurs questions de Diophante qui s'ont par ailleurs été résolues*, Paris, 1691.

³ In his *Traité d'algèbre* (1698), Rolle gives the "method of equations," but a example of an equation

$$f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n = 0$$

being understood the equation obtained by multiplying the terms of the original equation by the terms of the progression 0, 1, 2, ..., n and dividing both sides by x ; in other words the equation $f'(x)=0$. The theorem on which the method of equations is based is not Rolle's theorem but a corollary to it, viz., between two consecutive real roots of $f(x)=0$ there cannot be more than one real root of $f'(x)=0$.

nearly the same form by Lagrange's first theorem the remainder in Taylor's expansion for $f(x+p)$ after n terms and then put $n=1$. For this the 'Principles' recognize the two inequalities

$$\left. \begin{aligned} \phi(x+p) - \phi(x) - p\phi'(x+p) &> 0, \\ \phi(x+p) - 2(p-x)\phi'(x+p) &< 0 \end{aligned} \right\} (0 < p \leq h) \quad (2)$$

where $\phi(x+p)$ and $\phi'(x+p)$ are respectively the minimum and maximum of $\phi(x)$ for $0 \leq x \leq h$ as expressed in the mean value theorem. The inequalities appear in Lagrange's *Théorie des fonctions algébriques* in a generalized form and in the new form in his *Leçons sur le calcul différentiel* (1808) and in Ampère's paper. Both latter are quoted pointedly by the modern under-revues. Lagrange's paper of 1797, the same year Cauchy gives this proposition in his *Leçons* and in his extension form (1826) as a corollary. Perhaps the first to bring together the theorem of mean value and Lagrange's form was Cauchy himself when he derived the theorem of mean value from his own form given by Stieltjes in his *Sur la détermination des limites des intégrales*. Stieltjes found it however implicit in the pages of Bolzano in the proof of the theorem under which (1) first appears in history down to the first form in the works of the writers and followed closely. The present attempt is made for the validity of (1) is satisfactory first of all and is first given with care by Darboux in his lectures² at Paris during the years 1871-1872. Darboux's formulation and proof is contained in Prof. G. Darboux's *Leçons* on both Bolzano's proof and Jordan's proof as given in the first edition of *Leçons d'analyse* Vol. I of 1892. The conditions for the validity of (1) were made less restrictive by Prof. W. H. Young and Dr. J. C. Young³ and still less restrictive by Dr. A. N. S. Singh⁴.

I proceed now to give a brief historical account of the investigations of θ which seems to have been first started by Cauchy and his school.

¹ 'Sur les développées Tayloriennes des séries' (*Philosophical Magazine*, Series 3, Vol. 3, 1800, pp. 455-470, especially pp. 464-467).

² Introduced in 1875 in the shape of the book *Fondements généraux de la théorie des variables réelles*.

³ *Nouvelles Annales* for 1914, pp. 45-47, 153-154, 252-256, 453-454. Jordan's proof referred to is the one which appeared in his *Leçons sur le calcul différentiel* of 1862.

⁴ On derivatives and the question of the mean. *Quarterly Journal of Mathematics*, Vol. XI, 1904, pp. 1-26.

⁵ On the mean value theorem of the differential calculus. *Bulletin of the London Mathematical Society*, Vol. XIX, 1914, pp. 48-49.

probably began with the first example, $f(x) = x^2 + 1$, $f'(x) = 2x$, and $f(1) = 2$. It is a pity that the next example, $f(x) = x^3$, was what came next. In 1884, I noticed a slight variation in the example given by Mr. John North in 1880. But the first important example, the function $f(x) = x^2 + 1$, was the one that was used. The name of the author of this example, S. K. Law, is now forgotten. The works of these five authors have appeared only in the last nine years.

5

The various important forms of the mean-value theorem apart from the mean-value theorem for the function $f(x) = x^2 + 1$ may be said to be a collection of theorems for the function $f(x) = x^2 + 1$, back to the first example, $f(x) = x^2 + 1$, and the function $f(x) = x^2 + 1$.

If $f(x)$ is a function of x which remains constant for x between a and b , and which for x between a and b has a definite limit, then if $f(x) = f(a) + f(b)$, the value of $f(x)$ will be at least $f(a)$.

$$f(x+h) - f(x) = hf'(x+h).$$

6. The quantity $f(x)$ is between 0 and 1.

If $f(x)$ is a function of x (1863). The mean-value theorem is not applicable anywhere by the first theorem, but the range of the function is between 0 and 1, and the value of the function is what we shall find a theorem.

If $f(x)$ is a function of x (The Math. Zet. Bd. 1, p. 27).

On the mean-value theorem (The Math. Zet. Bd. 1, p. 27).

On the mean-value theorem (The Math. Zet. Bd. 1, p. 27). (Bulletin of the Cal. Math. Soc., Vol. 10, pp. 143-146).

On the mean-value theorem (The Math. Zet. Bd. 1, p. 27). (The Math. Zet. Bd. 1, p. 27).

Hayashi, Science Reports, Tokyo Imp. Univ., 1921.

Sakurai, Science Reports, Tokyo Imp. Univ., 1921. (The Math. Zet. Bd. 1, p. 27).

On the mean-value theorem (The Math. Zet. Bd. 1, p. 27). (The Math. Zet. Bd. 1, p. 27).

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neither the number a nor $a + \theta h$ necessarily lies between a_0 and $a_0 + h$. Hence, as a particular case of the theorem above, we have the following: If $f(x)$ is a function of x which is differentiable at a then $f(a + \theta h) - f(a) = \theta h f'(a + \theta h)$, $0 < \theta < 1$.

$$\phi(b) - \phi(a) = (b-a) \phi'(a + \theta b - a), \quad 0 < \theta < 1$$

III. Let x_0 and x_1 be real numbers. If x_0 and x_1 are not equal, choose $x_0 < x_1$ or $x_1 < x_0$ greater than x_0 and x_1 be a function of x which is continuous and differentiable for all values of x between x_0 and x_1 . Then

$$F(x_1) - F(x_0) = (x_1 - x_0) F'(x_0 + \theta x_1 - x_0),$$

θ being some proper positive fraction.

IV. The function $f(x)$ is continuous in M if and only if

V. If M is a point in \mathbb{R}^1 , then the function $f(x)$ is continuous at M and will admit a unique value for $x = M$ if and only if the differential coefficient for the same then

$$\phi(a + h) - \phi(a) = h \phi'(a + \theta h)$$

for some value of θ satisfying

VI. Moreover, if $f(x)$ is continuous at x_0 then as x approaches x_0 when x passes from x_0 to $x_0 + h$ then

$$f(x_0 + h) - f(x_0) = h f'(x_0 + \theta h),$$

θ being a number comprised between 0 and 1. The converse is also true: if $f(x)$ has a unique value for $x = x_0$ and the above is satisfied then $f(x)$ is continuous at x_0 .

VII. Consider $f(x)$ in \mathbb{R}^1 . When the function $f(x)$ has a unique value for $x = x_0$ and remains continuous at x_0 for which its differential coefficient $f'(x)$ from $x = x_0$ up to $x = x_0 + h$ then exists between the limits 0 and 1 is value of θ such that

$$f(x_0 + h) - f(x_0) = h f'(x_0 + \theta h).$$

- A Treatise on the Differential Calculus, p. 133.
- Topologie und Differentialrechnung, p. 72.
- The Differential Calculus, p. 67.
- Elements of Differential, p. 1.
- Leçons sur le Calcul Différentiel, p. 13.

existence of the differential coefficient were the two Youngs. Their generalization is the following:

XII If there is no distinction of right and left with regard to the derivatives of $f(x)$, then there is a point in the completely open interval (a, b) at which $f(x)$ has a differential coefficient and the value of that differential coefficient is precisely

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = f'(c) + o(h) \quad (0 < h < \epsilon)$$

The special case in which $f(x) = f(x) = 0$ is a special case of the general case. Bolle's theorem and the following special case of it are of interest. If $f(x)$ is a finite function which has maxima only at a point where $f(x) = 0$ and a further maximum throughout the closed interval $[a, b]$, and $f(x)$ is zero at the end points a, b , then there is a point c in the completely open interval (a, b) at which one of the upper derivatives is not positive and the other lower derivative is not negative.

$$\text{i.e., } f'(c) \leq 0 \leq f_-(c).$$

or the alternative inequality interchanging left and right,

$$f'(c) \leq 0 \leq f_-(c)$$

Young's theorem goes further than the Youngs and his generalization still more removes the conditions imposed on the derivatives. Singh's generalization, is the following

XIII If $f(x)$ be a continuous function defined on the closed interval $[a, b]$, such that

(i) there is no point within (a, b) at which one of the derivatives f_+ or f_- is progressively different with sufficient regularity (if one does not exist) even when the other does not exist, and $f(x)$ is not at each point within (a, b) the upper and lower derivatives on one side are equal to the upper and lower derivatives on the other side then there exists a point in the completely open interval (a, b) at which the differential coefficient exists and its value is equal to

$$f'(c) = f'(c)$$

Singh's is the following theorem. It is a theorem for derivatives: If $f(x)$ be a continuous function defined on the interval $[a, b]$ such that the upper

and ϕ are differentiable in the interval or are equal to the upper and lower derivatives at b . If $b = a$, then there is a point in the completely open interval (a, b) such that $f'(c) = \phi(c)$ and the value of ϕ at this point and its value is

$$\frac{f(b) - f(a)}{b - a}$$

§ 5. •

Let $f(x)$ be a function of x which is continuous in the interval $[a, b]$ and differentiable in the open interval (a, b) .

We shall now consider the expansion of the expression $f(a + h)$ in powers of h . Let $f(a) = A_0$. Then $f(a + h) = A_0 + A_1 h + A_2 h^2 + \dots$. The coefficients A_1, A_2, \dots are functions of a and f . The expansion is valid for h small enough so that $a + h$ lies in the interval (a, b) . The expansion is unique.

$$\theta = \frac{1}{2} + hA_1 + h^2A_2 + \dots + h^nA_n + \dots \text{ to infinity}$$

Let A_1, A_2, A_3, \dots be the derivatives of f at a . Then $A_1 = f'(a)$, $A_2 = \frac{1}{2}f''(a)$, $A_3 = \frac{1}{6}f'''(a)$, $A_4 = \frac{1}{24}f^{(4)}(a)$, $A_5 = \frac{1}{120}f^{(5)}(a)$, $A_6 = \frac{1}{720}f^{(6)}(a)$, $A_7 = \frac{1}{5040}f^{(7)}(a)$, $A_8 = \frac{1}{40320}f^{(8)}(a)$, $A_9 = \frac{1}{362880}f^{(9)}(a)$, $A_{10} = \frac{1}{3628800}f^{(10)}(a)$, $A_{11} = \frac{1}{39916800}f^{(11)}(a)$, $A_{12} = \frac{1}{479001600}f^{(12)}(a)$, $A_{13} = \frac{1}{6250803200}f^{(13)}(a)$, $A_{14} = \frac{1}{84458880000}f^{(14)}(a)$, $A_{15} = \frac{1}{1216680960000}f^{(15)}(a)$, $A_{16} = \frac{1}{18250214400000}f^{(16)}(a)$, $A_{17} = \frac{1}{283753216000000}f^{(17)}(a)$, $A_{18} = \frac{1}{4814169600000000}f^{(18)}(a)$, $A_{19} = \frac{1}{81902784000000000}f^{(19)}(a)$, $A_{20} = \frac{1}{1418860800000000000}f^{(20)}(a)$, $A_{21} = \frac{1}{25417344000000000000}f^{(21)}(a)$, $A_{22} = \frac{1}{453532800000000000000}f^{(22)}(a)$, $A_{23} = \frac{1}{8190278400000000000000}f^{(23)}(a)$, $A_{24} = 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$A_{130} = \frac{1}{111872000}f^{(130)}(a)$, $A_{131} = \frac{1}{213248000}f^{(131)}(a)$, $A_{132} = \frac{1}{405888000}f^{(132)}(a)$, $A_{133} = \frac{1}{76800}f^{(133)}(a)$, $A_{134} = \frac{1}{144000}f^{(134)}(a)$, $A_{135} = \frac{1}{2700}f^{(135)}(a)$, $A_{136} = \frac{1}{504000}f^{(136)}(a)$, $A_{137} = \frac{1}{93326400}f^{(137)}(a)$, $A_{138} = \frac{1}{171475200}f^{(138)}(a)$, $A_{139} = \frac{1}{314425600}f^{(139)}(a)$, $A_{140} = \frac{1}{579456000}f^{(140)}(a)$, $A_{141} = \frac{1}{1073744000}f^{(141)}(a)$, $A_{142} = \frac{1}{19958400}f^{(142)}(a)$, $A_{143} = \frac{1}{37488000}f^{(143)}(a)$, $A_{144} = \frac{1}{70329600}f^{(144)}(a)$, $A_{145} = \frac{1}{131788800}f^{(145)}(a)$, $A_{146} = \frac{1}{248832000}f^{(146)}(a)$, $A_{147} = \frac{1}{4684800}f^{(147)}(a)$, $A_{148} = \frac{1}{8812800}f^{(148)}(a)$, $A_{149} = \frac{1}{16588800}f^{(149)}(a)$, $A_{150} = \frac{1}{31459200000$

function θ in \mathcal{H} in order that these should not be responsible for the

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[illegible]

11. If f is a vector field defined on a region R in the xy -plane, and h and p are the functions defined by the equations

3) Let a and b be positive reals, $1 \leq a \leq b$ and also $1 \leq c \leq h \leq a$. Further let f be continuous on $[c, b]$ and differentiable in (c, b) . Then for the interval (c, b) and for every $\epsilon > 0$ there exists a δ such that for any value the norm holds. If u is a product of c and h then it is $c \cdot h$ and for most of the f it is $f(b) + \epsilon$ with $\epsilon > 0$.

10) With the assumptions made in 9) and $f(x, t) = f(x)$ and $f(x)$ independent only on x then all the functions $u_i(x, t)$ are at any t equal to $f(x)$ for the whole x and the value theorem is

$$\theta = \frac{1}{ah} \log \frac{e^{ah} - 1}{ah}$$

and the appropriate polynomial $f(x) = x^2 + \beta x + \gamma$, with $\beta, \gamma \in \mathbb{F}_q$.

In a similar manner, upon (2) June 10, 1943, the same day as the preceding day, the following information was obtained:

4. ϕ is a function which assigns to the set $\{a_1, \dots, a_n\}$ the value $\phi(a_1, \dots, a_n)$. V. T. 1906, pp. 17-18. In the discussion of set-valued functions the notion of confusion in the absence of the explicit isolation of $\phi(a_1, \dots, a_n)$ is used to study as a single-valued function of b .

* "On Rolle's function & its multiple-valued function" (*Journal of the London Mathematical Society*, Vol. X, 1932). See also his paper on "On the function $\phi(x)$ in the theory of Weierstrass's non-differentiable function" (*Journal of the London Mathematical Society*, Vol. X, 1932). For more on the mean value theorem for the case of a function differentiable on a set, see also his paper "On the function $\phi(x)$ in the theory of Weierstrass's non-differentiable function" (*Journal of the London Mathematical Society*, Vol. X, 1932).

(c) With the support of the lemma it is not difficult to show that if f is independent of h and depends only on x , then the mean value theorem for f exists which also possesses

$$\frac{df}{dx}.$$

In the above lemma in the same paper the following question is asked: what conditions must be satisfied for f as a function of x and h so that there shall exist a corresponding function f' satisfying the mean value theorem

$$f(x+h) = f(x) + h f'(x+\theta h)?$$

In treating this question, the hypothesis of the continuity and existence of

$$u_1 = \frac{\partial f}{\partial x_1}, \quad u_2 = \frac{\partial f}{\partial x_2}, \quad \dots, \quad u_n = \frac{\partial f}{\partial x_n}, \quad u_{11} = \frac{\partial^2 f}{\partial x_1^2}, \quad u_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

where $x_1 = x$ and $x_2 = x+h$.

§ 7.

12. In this section we shall introduce a new method of dealing with the mean value theorem, a number of miscellaneous theorems will be introduced which will be thus connected with the mean value theorem, for the latter is found to be a kind of generalization of the mean value theorem for a function of a given linear variable.

$$\begin{aligned} f(x+h) &= f(x) + h f'(x+\theta h) \\ f(x+h) &= f(x) + \frac{1}{2} f''(x+\theta h) h^2 \end{aligned} \quad (0 < \theta < 1)$$

In a different way known generalizations of the above theorem by Thompson and Pearson may be introduced to the mean value theorem.

The latter theorem of Thompson

$$\begin{vmatrix} f(x) & f(x+h) & f(x+\theta h) \\ f'(x) & f'(x) & f'(x) \\ f(x+\theta h) & f(x+\theta h) & f(x+\theta h) \end{vmatrix} = 0 \quad (0 < \theta < 1)$$

See also differential equations and integral equations. For $\theta = 1$, the above reduces to Cauchy's generalized mean value theorem.

constant μ given by $\mu = \mu(x, y, D)$ depending on x, y, D (Hayashi² and Takahashi³). If $\mu = 1$, $P(x, y, D) = 1$, the estimate (1.1) reduces to the estimate (1.2) that under certain conditions (for ϕ and f)

$$|f(x)| \leq C \int_0^h |f(x + u_1 h)| |f(x + u_2 h)| \cdots |f(x + u_n h)| P(x + u_1 h, \dots, x + u_n h) du_1 \cdots du_n \quad (1.3)$$

The theorems of Hayashi and Takahashi are complicated and involve many groups G_n such that $h \in G_n$ and $h \in G_n$ (Hayashi's theorem).

(ii) The next group of results is concerned with the relation between the function μ and the function μ_n (see § 2 in Takahashi's paper) or $\frac{h}{n}$ (see § 3 in Hayashi's paper) as the parameter n tends to infinity. In this case, the value of μ is independent of n . In the case of the ordinary μ (see § 4 in Hayashi's paper), Whitehead⁴ gave the expansion of μ in powers of h where the coefficients of h^k ($k \geq 1$) are the constant μ_k (see § 5 in Whitehead's paper). The function μ_k has been studied by Whitehead⁴ and the results (among others) have been extended by him.

(iii) In order to obtain a result satisfying Taylor's theorem with the remainder $r = O(h^{n+1})$, it is necessary to satisfy a condition on μ independent of n and h . It is necessary and sufficient that $f(x)$ be a polynomial of the $(n+1)$ th degree, and that $\theta_n \pi \frac{1}{n+1}$.

$$(b) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \mu_n(x) dx = \frac{1}{n+1} \int_0^h \mu_{n+1}(x) dx \quad \text{exists and is continuous for}$$

$0 < x < h$ and at the same time n tends to infinity.

$$(c) \quad \mu_n(x) = \frac{1}{n+1} \left\{ \frac{1}{n+2} - \frac{1}{2n+1} \right\} \cdot \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)} \quad \text{for}$$

certain conditions. The work of Whitehead in this connection is open to two serious criticisms as in the case of Hadamard's function θ .

But we follow the theorems for approximations (6) (see § 3 in Hayashi's paper and § 4 in Takahashi's paper) (pp. 61-64).

¹ The *Science Report of the Tokyo Imperial University*, Vol. 15, 1925, pp. 185-191.

² *Tokyo Imperial University*, Vol. 30, 1929, p. 133.

³ *Math. Z.*, 33, 1929, pp. 342-371.

⁴ *Math. Z.*, 33, 1929, pp. 372-399.

⁵ *Math. Z.*, 33, 1929, pp. 399-411.

⁶ *Tokyo Imperial University*, Vol. 29, 1928, p. 151.

Put $\phi = f(x, y)$ and let $\phi(x, y) = 0$ be the equation of the curve. One can find $\phi(x, y)$ and $\phi(x, y + h)$ and it has to be known for a fixed x that $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$. Taking $x = 0$ and $y = 0$ it follows that $\phi(0, y) = 0$ if and only if $y = 0$ or $y = 1/2$.

Let $\phi(x, y)$ be the function $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$. Then $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$.

$$f(x+h, y+h) = f(x, y)$$

(a) The result that

$$f(x+h, y+h) = f(x, y) + h_1 \phi_1(x, y) + h_2 \phi_2(x, y) + \dots + h_n \phi_n(x, y)$$

holds for $f(x, y) = \phi(x, y)$ if and only if $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$.

and $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$.

(b) The result that

$$f(x+h, y+h) = f(x, y) + h_1 \phi_1(x, y) + h_2 \phi_2(x, y) + \dots + h_n \phi_n(x, y)$$

holds for $f(x, y) = \phi(x, y)$ if and only if $\phi(x, y) = 0$ if and only if $y = 0$ or $y = 1/2$ and $\phi(x, y + h) = 0$ if and only if $y = 0$ or $y = 1/2 + h$.

$$x = x_0 + ht, \quad y = y_0 + ht.$$

$$f(x, y) = f(x_0 + ht, y_0 + ht) = f(x_0, y_0) + h_1 \phi_1(x_0, y_0) + h_2 \phi_2(x_0, y_0) + \dots + h_n \phi_n(x_0, y_0)$$

$$f(x, y) = f(x_0 + ht, y_0 + ht) = f(x_0, y_0) + h_1 \phi_1(x_0, y_0) + h_2 \phi_2(x_0, y_0) + \dots + h_n \phi_n(x_0, y_0)$$

$$f(x, y) = f(x_0 + ht, y_0 + ht)$$

$$f(x, y) = f(x_0 + ht, y_0 + ht) = f(x_0, y_0) + h_1 \phi_1(x_0, y_0) + h_2 \phi_2(x_0, y_0) + \dots + h_n \phi_n(x_0, y_0)$$

SECOND LECTURE

PROOF OF THE MEAN VALUE THEOREM

11

Let $f(x)$ be a function which is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) . We shall prove that there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. This is the Mean Value Theorem. The proof is based on the fact that a continuous function on a closed interval attains its maximum and minimum values. As a consequence, the function $f(x)$ is bounded on $[a, b]$. Hence, it is continuous on $[a, b]$ and differentiable on (a, b) . (almost word by word)

Let $f(x)$ be a function which is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) . We shall prove that there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. This is the Mean Value Theorem. The proof is based on the fact that a continuous function on a closed interval attains its maximum and minimum values. As a consequence, the function $f(x)$ is bounded on $[a, b]$. Hence, it is continuous on $[a, b]$ and differentiable on (a, b) .

$$\frac{f(b) - f(a)}{b - a} = \frac{f(x) - f(a)}{x - a}$$

x , being a value comprised between a and b .

In fact, the ratio

$$\frac{f(x) - f(a)}{x - a}$$

has by hypothesis a limit as x approaches a and is equal to $f'(a)$. Hence, we have

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a). \quad (1)$$

Let us denote by $\phi(x)$ the function defined by the expression

$$\phi(x) = \frac{f(x) - f(a)}{x - a} - f'(a). \quad (2)$$

then we should have, because of the equality (1),

$$\lim_{x \rightarrow a} \phi(x) = 0.$$

so that $\phi(x)$ vanishes for $x = a$ and for $x = b$.



One has therefore

$$\frac{f(X) - f(x_0)}{X - x_0} = f'(x_0), \quad (1)$$

as was announced.

We have supposed that $X > x_0$ but the preceding theorem is not changed with the modification of the other $X < x_0$ in place of this hypothesis.

If one puts

$$X = x_0 + h$$

the quantity x comprised between x_0 and $x_0 + h$ can be represented by $x_0 + \theta h$, θ being a quantity comprised between 0 and 1. Then, one can write

$$f(x_0 + h) - f(x_0) = hf'(x_0 + \theta h).$$

It is necessary to make a few remarks about the above proof.

(i) Although the function $f(x)$ is continuous everywhere, it may not be the values of x comprised between the points x_0 and X and not at the points themselves. The continuity at the points x_0 and X is not sufficient to ensure the existence of the derivative at these points. The proof of the theorem shall obtain its upper and lower bound at a finite point x because the δ and ϵ limits will not necessarily be large.

(ii) The proof does not assume the existence of $f'(x_0)$ or X .

(iii) The proof does not assume either the continuity of $f(x)$ anywhere or its finiteness.

(iv) The proof is a simple one, as pointed out by the above, because of the statement: "It is sufficient that δ be sufficiently small for the values of x comprised between x_0 and X to correspond to a function $f(x)$ increasing by taking positive values or decreasing by taking negative values." This is obtained from x_0 and X in a very convenient manner.

If we take $\delta(x)$ to be $x - x_0$, then $\delta(x) = x - x_0$ and $\epsilon(x) = x - x_0$.

It is not possible to say that $\delta(x)$ begins to decrease as x increases because in any interval over a small $\delta(x)$ for x may have a number of fluctuations.

§ 9

Let $\lambda = \lambda(x)$ be a continuous function of x on the interval $[x_0, x_1]$ and let f be a function of x on the same interval. Let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$.

Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$.

$$f(x_1) - f(x_0) = (x_1 - x_0) f'(x_0)$$

θ being some proper positive fraction

Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$.

Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$.

$$f(x_1) - f(x_0) = (x_1 - x_0) f'(x_0)$$

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(x_1) \quad \left. \begin{array}{l} \vdots \\ \vdots \end{array} \right\} \quad (1)$$

$$f(x_n) - f(x_{n-1}) = (x_n - x_{n-1}) f'(x_{n-1})$$

whence, adding the first n equations, we find the sum of the first n terms of the series $f(x_1) - f(x_0), f(x_2) - f(x_1), \dots, f(x_n) - f(x_{n-1})$ is equal to the sum of the first n terms of the series $(x_1 - x_0) f'(x_0), (x_2 - x_1) f'(x_1), \dots, (x_n - x_{n-1}) f'(x_{n-1})$.

$$f(x_n) - f(x_0) = (x_n - x_0) f'(x_0)$$

which is a proper positive fraction

§ 10

Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$. Let $f(x)$ be a function of x on the interval $[x_0, x_1]$ and let $f(x_0) = f_0$ and $f(x_1) = f_1$.

If the function $f(x)$ is continuous on the interval $[a, b]$ and if $f(a) \neq f(b)$, then there exists a point ξ in the interval (a, b) such that $f(\xi) = \frac{f(a) + f(b)}{2}$.

$$\frac{f(X) - f(x_0)}{X - x_0} = \dots (4)$$

If the function $f(x)$ is continuous on the interval $[a, b]$ and if $f(a) \neq f(b)$, then there exists a point ξ in the interval (a, b) such that $f(\xi) = \frac{f(a) + f(b)}{2}$.

$$f(x + \delta) - f(x)$$

is always greater than ϵ and that $f(x) > f(a)$. If between the points x_0 and x_1 there is a point ξ such that $f(\xi) = \frac{f(x_0) + f(x_1)}{2}$,

$$x_1 - x_0 = \delta$$

so as to divide the difference $X - x_0$ into n parts

$$x_1 - x_0 = x_2 - x_1 = \dots = X - x_0$$

which being all of the same length, the number of parts is $\frac{X - x_0}{\delta}$. The fractions

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}, \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \dots, \frac{f(X) - f(x_0)}{X - x_0}$$

being all of the same length, the first part of the interval $[x_0, x_1]$ is equal to the second part of the interval $[x_1, x_2]$, and so on. Let A be the quantity $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$ and let B be the quantity $\frac{f(X) - f(x_0)}{X - x_0}$. Moreover, the fractions (5) having denominators of the same length, the sum of the numerators is the sum of the lengths of the intervals. This is a fraction that is less than one and the whole of the sum of those which are considered.

The expression (5) which is the mean value of the function $f(x)$ is therefore enclosed between the limits A and B , and as the number n is however small, the number n is large, it follows that the expression (5) shall be comprised between A and B .

For this, if the differential coefficient $f'(x)$ is not constant between the limits $x = x_0$ and $x = X$, in passing from x_0 to X , the function shall vary in such a manner as to approach a value



between A and B and it takes on every value in the intermediate values. Therefore every intermediate quantity y between A and B shall be a value of $f(x)$ corresponding to a value of x taken between a and b and $X = x$, or what amounts to the same thing $f(X) = y$ or $f(x) = y$.

$$x_0 + \delta h = x_0 + \theta (X - x_0).$$

θ denoting a number less than 1."

It is at this necessary to make the following remark in the next proof.

We shall by way of example start with $x_0 = x$, $x_1 = x + h$ and $f(x_1) = f(x + h) = y$ and $f(x) = f(x) = y$ in the corollary. This is not necessary for the proof that

$$\frac{f(X) - f(x_0)}{X - x_0}$$

lies between A and B .

(b) To carry the proof I take x_0 and x_1 as two arbitrary quantities that for any value y between A and B there is a value of x such that $f(x) = y$. The ratio $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$ is greater than A and less than

B and n_k is a less than ϵ assuming merely that

we can take $f(x) = f(x_0)$ for every value of x . To see this we refer to the continuity of $f(x)$.

Take $x_0 = x$, $x_1 = x + h$ and $f(x_1) = f(x + h) = y$ and $f(x) = f(x) = y$ for every value of x and h the ratio included. But because of the discontinuity of $f(x)$ at x_0 , $f(x_1) = f(x + h)$ does not uniformly tend to

$f(x_0)$ as h approaches 0 as an endpoint. For taking $x_1 =$

$$x_0 + \frac{1}{2m} \quad x_1 = x_0 + \frac{1}{2m} \quad x_2 = x_0 + \frac{1}{2m} \quad \dots \quad x_m = x_0 + \frac{1}{2m} \quad \text{where } m \text{ is integral. } (f(x_1) - f(x_0)) = 0,$$

$f(x_1) = 1$ so that the difference $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1 - 0}{\frac{1}{2m}} = 2m$ is numerically equal to 1 and it is not a vanishing quantity with increasing m .

2) Another fairly careful proof is De Morgan's and is set proved below. Let there be two limits A and $B = h$ such that neither for some

not between them, so that no any number, ϵ , can be found. Then for $\log x$, from $x = 2$ to $x = 1$ there is no singular value of $\log x$, and $\log 1$ is not of them singular. We have now $P' = \frac{1}{x^2}$ and $\log x$ with x whatever the value of x may be, is not singular. Consequently P and $\log x$ will remain continuous even to $x = 1$ when x diminishes, and should vary in any manner between a and $a + h$. Thus for instance x and $\log x$ are continuous even to $x = 1$ while Δx diminishes, and should vary in any manner between a and $a + h$.

Let us suppose $\phi(x)$ to be the n^{th} part of $a + x$ that ϕ makes without limit as n increases without limit. If ϕ is a function of x and Δx be limited by $f(x) = \phi(x)$ and we then have

$$\phi(x + \Delta x) - \phi(x) = \phi'(x) + f(x, \Delta x);$$

now we let take n successively $2, 3, 4, \dots$ for x and we come to have $\phi(x + n \Delta x) - \phi(x) = n f(x, \Delta x)$ in the numerator, which will give the following set of equations (n is number):—

$$\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \phi'(x) + f(x, \Delta x),$$

$$\frac{\phi(x + 2 \Delta x) - \phi(x)}{2 \Delta x} = \phi'(x) + f(x, 2 \Delta x),$$

$$\frac{\phi(x + 3 \Delta x) - \phi(x)}{3 \Delta x} = \phi'(x) + f(x, 3 \Delta x),$$

..

$$\frac{\phi(x + n \Delta x) - \phi(x)}{n \Delta x} = \phi'(x) + f(x, n \Delta x) + f(x + n \Delta x, \Delta x)$$

$$\frac{\phi(x + n \Delta x) - \phi(x + n \Delta x - 1 \Delta x)}{\Delta x} = \phi'(x + n \Delta x - 1 \Delta x) + f(x + n \Delta x - 1 \Delta x, \Delta x)$$

Form the fraction which has the sum of the numerators of the preceding for its numerator and the sum of the denominators for its denominator it being clear that all the denominators have the same sign

* But the definition of the expression $\phi(x)$ does not seem clear p. 14 of De Morgan's book. The definition is, however, not clear. Perhaps, all that can be safely assumed is that De Morgan meant $\phi(x)$ to have a singular value at x if $\phi(x) = \phi(x + \Delta x)$ had a discontinuity at x .

* P is said to be a continuous if it tends to $\phi(a)$ as x tends to zero, $\phi(x) = \phi(x + \Delta x)$.

This gives

$$\phi(x) - \phi(x_0) = \phi'(x_0 + k\Delta x) \Delta x \quad \text{or} \quad \frac{\phi(x) - \phi(x_0)}{\Delta x} = \phi'(x_0 + k\Delta x)$$

$$\text{or } \frac{\phi(x + k\Delta x) - \phi(x)}{\Delta x} = \phi'(x + k\Delta x) \quad \text{or} \quad \frac{\phi(x) - \phi(x_0)}{\Delta x} = \phi'(x_0 + k\Delta x)$$

Let us now return to the question of the greatest and least values of ϕ between x_0 and x_1 . Let ϕ have a greatest value at x and let ϕ have a least value at x_0 . Then $\phi(x) - \phi(x_0) = \phi'(x_0 + k\Delta x) \Delta x$ and so the formula

$$\phi'(x_0 + k\Delta x) = \phi'(x_0 + k\Delta x, \Delta x)$$

holds for all Δx between x_0 and x_1 and for all k between 0 and 1. Let x_0 and x_1 be the values of x and k which give the greatest and least values of ϕ between x_0 and x_1 . Then $\phi(x_0) - \phi(x_1) = \phi'(x_0 + k\Delta x) \Delta x$ and so the formula

$$\phi'(x_0 + k\Delta x) = \phi'(x_0 + k\Delta x, \Delta x) \quad \text{and} \quad \phi'(x_0 + k\Delta x) = \phi'(x_0 + k\Delta x, \Delta x)$$

hold for all Δx between x_0 and x_1 and for all k between 0 and 1. Now, if ϕ has a greatest value at x_0 and a least value at x_1 , then $\phi(x_0) - \phi(x_1) = \phi'(x_0 + k\Delta x) \Delta x$ and so the formula

$$\phi'(x_0 + k\Delta x) = \phi'(x_0 + k\Delta x, \Delta x) \quad \text{lies between } \phi'(x_0) \text{ and } \phi'(x_1)$$

Let ϕ be a function of x which is continuous on the interval $[x_0, x_1]$ and let ϕ have a greatest value at x_0 and a least value at x_1 . Then $\phi(x_0) - \phi(x_1) = \phi'(x_0 + k\Delta x) \Delta x$ and so the formula

$\phi'(t)$ between $\phi'(a)$ and $\phi'(b)$, and since $a + \theta h$ here is between a and b , ...

$$\frac{\phi(a+h) - \phi(a)}{h} = \phi'(a + \theta h).$$

§ 12

21. The following proof of Lagrange's Theorem may be considered to be a particular case of the more generalised mean value theorem of Cauchy to be treated below. Let $f(x)$ and $F(x)$ be two real functions which vanish for $x = a$, and which remain continuous between the limits $x = a$ and $x = X$. Let us suppose that the differential coefficient $F'(x)$ does not change its sign between the limits in question. If one calls A the least and B the greatest of the values which the ratio

$$\frac{f'(x)}{F'(x)}$$

receives in this interval, then the fraction

$$\frac{f(x)}{F(x)}$$

shall also remain comprised between the two limits A and B .

Proof. — Because we shall have, by hypothesis, for all the values of x enclosed between the limits a and X ,

$$\frac{f'(x)}{F'(x)} - A > 0, \quad \frac{f'(x)}{F'(x)} - B < 0$$

and because the differential coefficient $F'(x)$ does not change its sign between these limits, one may affirm that in this interval one of the products

$$F'(x) \left\{ \frac{f(x)}{F(x)} - A \right\} = f'(x) - AF'(x),$$

$$F'(x) \left\{ \frac{f(x)}{F(x)} - B \right\} = f'(x) - BF'(x)$$

remains positive or negative. (Cauchy, Series 2, t. IV, pp. 402-412). This proof was first given in the *Addition to the Résumé* (Cauchy, Series 2, t. IV, pp. 242-261).

shall be of the same sign as the corresponding ΔF . Moreover these products are reciprocals, so that the two functions

$$f(x) - \Delta F(x), f(x) - \Delta F(x)$$

have the same sign as ΔF and ΔF respectively. They vanish only at $x = X$, where $\Delta F = 0$. They vanish nowhere else.

$$f(X) - \Delta F(X), f(X) - \Delta F(X)$$

are the same as ΔF and ΔF respectively. They vanish only at $x = X$, where $\Delta F = 0$. They vanish nowhere else.

$$\frac{f(X) - \Delta F(X)}{f(X) - \Delta F(X)} = A, \frac{f(X) - \Delta F(X)}{f(X) - \Delta F(X)} = B$$

Insert the values A and B in the equation $f(x) - \Delta F(x) = A \Delta F(x)$.

If the function $f(x)$ is continuous at $x = X$, then when one passes

from $x = X$ to $x = X + \Delta x$, the ratio $\frac{f(x) - \Delta F(x)}{f(x) - \Delta F(x)}$ shall vary in such a manner

as to remain always comprised between the two values A and B and to

take on every value between A and B . Therefore the fraction $\frac{f(x) - \Delta F(x)}{f(x) - \Delta F(x)}$ shall be a

value of the ratio $\frac{f(x) - \Delta F(x)}{f(x) - \Delta F(x)}$ corresponding to a value of x lying between

the limits X and $X + \Delta x$. Putting $x = X + \theta \Delta x$

$$\frac{f(X + \theta \Delta x) - \Delta F(X + \theta \Delta x)}{f(X + \theta \Delta x) - \Delta F(X + \theta \Delta x)} = A + \theta(B - A)$$

Let $\theta = 0$, $f(X) - \Delta F(X)$ and $\theta = 1$, $f(X + \Delta x) - \Delta F(X + \Delta x)$ have the same sign

2. Before giving the proof of the theorem, let us say a few words about each of the following proofs: Lagrange's, Cauchy's, Heine's.

§ 15.

25* In 1801 D. Pompei¹ gave a proof of the mean value theorem which differs from all the other proofs in this respect, that it is not based on Rolle's theorem, which is very easy to be proved, but on a previously established lemma the mean value theorem is deduced from it. I shall now proceed to give this proof of Pompei.

Let a function $y = f(x)$ be continuous in an interval (not necessarily closed) and let $f(x)$ have a unique value for every point x of that interval (theorems 1, 2, 3, 4). Further let a and b be two points in the given interval, $a < b$, and let a_1 be a point between a and b . Then from the ratio

$$R(a, b) = \frac{f(b) - f(a)}{b - a}$$

The question is to prove that there is at least a point between a and b the differential coefficient at which has precisely the value $R(a, b)$.

Now let

$$a_1 = \frac{1}{2}(a + b)$$

Then we have

$$R(a, b) = \frac{1}{2} \left\{ \frac{f(a) - f(a_1)}{a - a_1} + \frac{f(a_1) - f(b)}{a_1 - b} \right\}$$

or, briefly,

$$R(a, b) = \frac{1}{2} \{ R(a, a_1) + R(a_1, b) \}. \quad \dots (1)$$

The cases are all the same as in the right side of the above inequality are equal in which case we may consider any one of them. If the two ratios are not equal to that is, if they are greater and the other less, than $R(a, b)$.

(a) Taking up the second case first consider the function defined by

$$R(x, a_1) = \frac{f(x) - f(a_1)}{x - a_1}, \quad x \neq a_1$$

$$R(x, a_1) = f'(x) \equiv 0, \quad x = a_1$$

Then obviously for every value of x in the interval (a, b) other than a_1 , $R(x, a_1)$ is continuous in x a_1 's because of $f(x, a_1)$ being defined to be $f(x_1)$ it is continuous at a_1 . Thus $R(x, a_1)$ is continuous for every

¹ C'est le théorème des accroissements finis. Annales 5 scientifiques de l'Université de Jassy, 1808.

value of c in I_2 by the only thing needed further, openness of the intervals of R (i.e., $a_1 < a_2$ and $b_1 < b_2$), between I_1 and I_2 and $f(a_1) \neq f(b_1)$. Thus the continuous function f must take the value $f(a_1)$ for some value b_1 of x intermediate between a_1 and b_1 , so that

$$R(b_1, a_1), (i.e., R(a_1, b_1)) = R(a, b). \quad \dots (2)$$

If the point b_1 coincides with a_1 , then the theorem is proved as we will have

$$f(a_1) = R(a, b).$$

In any case

$$|a_1 - b_1| \leq \frac{1}{2} |a - b|.$$

If a_1 and b_1 are unequal, take

$$a_2 = \frac{1}{2}(a_1 + b_1)$$

Then proceeding as in the case of $R(a, b)$ it is proved that between a_1 and b_1 a point b_2 exists such that

$$R(a_2, b_2) = R(a_1, b_1). \quad (3)$$

where

$$|a_2 - b_2| \leq \frac{1}{2} |a_1 - b_1|$$

We continue this reasoning indefinitely if none of the points b_k coincides with the corresponding point a_k .

We assume that a set of intervals $\{I_k\}$ possessing the following properties:

(1) the interval I_k (i.e., $a_k - b_k$) is inside the interval I_{k-1} and added on longer at the end of I_{k-1} than half of the length of I_{k-1} so that the length of I_k (i.e., $a_k - b_k$) $\leq \frac{1}{2} |a - b|$

(2) $R(a_k, b_k) = R(a, b)$ for every value of the integer k .

Therefore I_k tends to a limiting interval as k is indefinitely increased, but when a_k and b_k tend to c from the definition of f , it is obvious that $R(a_k, b_k)$ tends to $f(c)$.

Thus it is proved that

$$\frac{f(a) - f(b)}{a - b} = f$$

where

$$a < c < b$$

(3) If, in (1), $R(a_{k+1}, b_{k+1})$ and $R(a_k, b_k)$ are equal then one end of I_{k+1} above, we have

$$R(a_1, b) = R(a, b)$$

so that $b_1 = b$ with

$$|a_1 - b_1| = \frac{1}{2} (a - b).$$

For, when $x \rightarrow 0$, we get the same result in the end as there can the mean value theorem is proved.

Criticism of Pompeiu's proof.

In order that Pompeiu's proof should be correct, the function $f(x)$ must be continuous at $x = 0$ and $f(x)$ have the same limit as $x \rightarrow 0$. It does not necessarily follow that $\lim_{x \rightarrow 0} f(x) = f(0)$ tends to

$$f(0) \text{ as } x \rightarrow 0, \text{ but } f(x) = \frac{1}{x} \text{ is a function which does not tend to } f(0) \text{ as } x \rightarrow 0.$$

$$y = \frac{1}{x} + \frac{1}{x^2}$$

Here, $y \rightarrow \infty$ as $x \rightarrow 0$. But $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$ does not tend to $f(0)$.

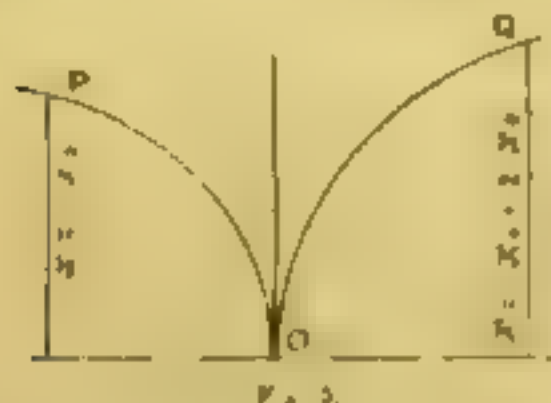
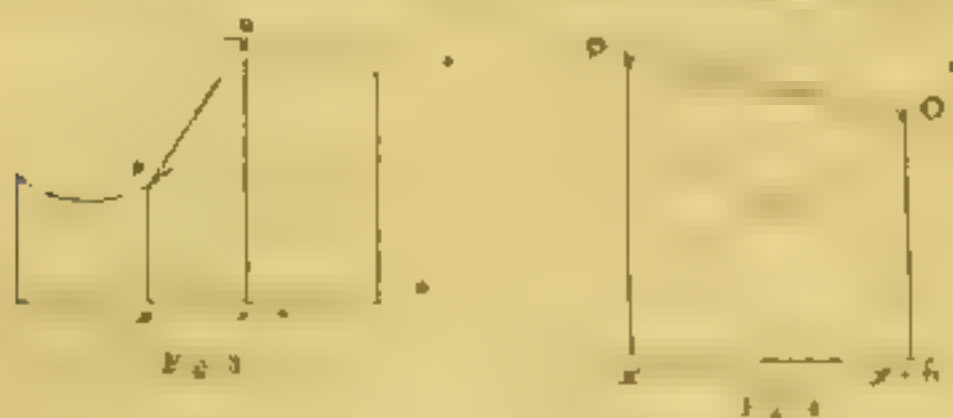
i.e., 0, but to $-\frac{\pi}{2}$.

§ 16.

It is possible now to consider the geometrical interpretation of the mean value theorem. If we assume that (21) admits of a graph for the interval $[a, b]$, an assumption which involves departure from regularity only at a finite number of points, then the geometrical interpretation of the theorem is as follows. To the chord joining any two points P and Q on the graph, say PQ , there exists at least one tangent at an interior point p which is parallel to the chord. The truth of this interpretation is illustrated by the figures 1 and 2, the inflexions of $f(x)$ at a point in the second figure not invalidating the theorem.



That the theorem may be illustrated is illustrated by the figures 3, 4.



In fig. 3 there is a discontinuity in $f(x)$ between P and Q; in fig. 4 $f(x)$ is non-existent at a point between P and Q; in fig. 5 at O between P and Q there is a cusp but $f(x)$ is non-existent there.

Other geometrical interpretations may be obtained by treating $f(x)$ as the area of the curve bounded by $y = f(x)$ and the x -axis or by treating $f(x)$ as a volume.

6.17

27. I will conclude this lecture by reviewing a number of important theorems from the mean-value theorem in Darboux's form, i.e., with the assumptions that $f(x)$ is finite and continuous in the interval a, b the ends being included, and that $f(x)$ is existent at every point in the whole of the interval the ends being excluded.

(i) Theorem A¹. If $f(x) = \text{const.}$ at every point inside (a, b) then $f(x)$ is constant in the whole of the interval.

¹ Although it is well known that $f(x)$ is zero in an interval if $f(x)$ is constant & that $f(x) > 0$ in an interval if at every point in that interval $f(x)$ is increasing, the geometric propositions are not self-evident. The simplest logical proofs of these converse propositions are, as given here by applying the mean value theorem.

Theorem 1. Let x_1, x_2 be any two points in (a, b) then by the mean-value theorem

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(\xi) \quad \dots (1)$$

where ξ is a point between x_1 and x_2 . If $f'(x) = 0$ by the hypothesis. Therefore $f(x_2) = f(x_1)$, thus the theorem is proved.

(i) **Theorem II.** If $f'(x) > 0$ at every point inside (a, b) then $f(x)$ is a constantly increasing function. If $f'(x) < 0$ at every point then $f(x)$ is a constantly decreasing function.

Proof. Applying (1) since $x_2 > x_1$ and $x_2 - x_1 > 0$. Then $f(x_2) > f(x_1)$ and it is proved that $f(x)$ is always increased with x . Similarly if $f'(x) < 0$ at every point of (a, b) that $f(x)$ is constantly decreased with increasing x .

(j) **Theorem III.** $f'(x) \neq 0$ in an interval (a, b) cannot be infinite.
Proof. Again applying (1),

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \text{ a finite quantity}$$

Therefore inside (x_1, x_2) there is at least one point ξ where $f'(x)$ is not infinite. Thus the theorem is proved.

(k) **Theorem IV.** $f'(x)$ does not become constant throughout an interval (a, b) unless $f(x)$ is constant everywhere in (a, b) that is $f(x) = c$.

Proof. For applying (1) $f(x_2) = f(x_1)$ is different from 0 and the theorem is proved.

(l) **Theorem V.** If at a point a in (a, b) the limit of $f'(x)$ exists then that limit will be the value of $f'(a)$ that is $f'(a) = \lim_{x \rightarrow a} f'(x)$ as long as a is inside (a, b) , is a or b .

Proof. Let x be an arbitrary point then by the mean value the theorem

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta h), \quad 0 < \theta < 1.$$

Therefore as h tends to 0, $a + \theta h$ tends to a consequently $f'(a + \theta h)$ tends to a definite limit say l by the hypothesis. But by the definition of the differential coefficient at a ,

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Thus it is proved that $f'(a) = l$. Similarly, the other cases can be dealt with.

(f) **Theorem F** If n satisfies the assumptions in (4) or (5) form of the mean value theorem $f(x) = \phi(x)$ and ϕ is then corresponding to an arbitrarily small quantity $\epsilon > 0$, ϵ is always positive, and a second quantity $\delta > 0$ and independent of x such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \delta, \quad \dots (2)$$

if $|h| < \epsilon$ whatever be the value of x in (a, b) .

Proof By the mean value theorem the left side of (2) is equal to $|f(x+\theta h) - f(x)|$. But by the hypothesis $f(x)$ is continuous in (a, b) and therefore uniformly continuous. Therefore corresponding to δ a quantity ϵ exists such that

$$|f(x_2) - f(x_1)| < \delta,$$

whenever x_1, x_2 and x_1 may have in (a, b) provided that $|x_2 - x_1| < \epsilon$.

Now $|\theta h| < |h|$; therefore

$$|f(x+\theta h) - f(x)| < \delta$$

if $|h| < \epsilon$.

7. **Theorem G** $f(x)$ takes every value between its upper and lower bounds in (a, b) in addition to the assumptions in (4) or (5) form of the mean value theorem it being assumed that $f(a+) = f(b-)$ exist.

Proof By Theorem F $f(x) = \phi(x)$ is either a continuous function $f(x)$ or it does not have the second kind of discontinuity; therefore, in any case, $\phi(x)$ takes every value between its upper and lower boundaries as required by Weierstrass's theorem about continuous functions and by Darboux's theorem about functions which have discontinuities of the second kind.

An alternative proof¹ is the following

Let U and L be the upper and lower boundaries of $f(x)$ in (a, b) . Then if $U' > U > L$ values, and x_1 of x exist such that in a neighbourhood of x_1 any $x_2 = x_1 + h$ (the h were arbitrary) if $f(x_1) < L$ and in a neighbourhood $(x_2 - h_2, x_2 + h_2)$ of x_2 the upper boundary is U' so that $f(x_1) < L + \epsilon_1$ and $f(x_2) > U' - \epsilon_2$ and ϵ_1, ϵ_2 being quantities greater than 0 but as small as we please.

¹ Given by Darboux in *Annales de l'École Normale Sup.* Series 2 Vol. 4 1870 p. 109.

² See Dini's *Calcolo Infinitesimale* p. 11 pp. 125.

Now consider the function

$$\phi(x) \equiv f(x) - Cx$$

in [a, b], with $h = x_2 - x_1 > 0$ sufficiently small, we have obviously

$$\lim_{h \rightarrow 0} \frac{\phi(x_1 + h) - \phi(x_1)}{h} < 0$$

$$\lim_{h \rightarrow 0} \frac{\phi(x_2 - h) - \phi(x_2)}{-h} > 0.$$

Hence there must exist a number $\delta > 0$ such that for values of h between x_1 and $x_1 + \delta$ we have $\phi(x) < \phi(x_1)$, and for values of h between $x_2 - \delta$ and x_2 (x_2 excluded) we shall have respectively the inequalities

$$\phi(x) - \phi(x_1) < 0,$$

$$\phi(x) - \phi(x_2) < 0$$

Therefore we do not need to find a number x_0 such that x_0 is a maximum of $\phi(x)$. Now for $x_0 + h$ and $x_0 - h$ (with h sufficiently small) there must exist values η_1 and η_2 where the mean value theorem is satisfied for h less than a certain value δ .

$$\phi(x_0 + h) - \phi(x_0) \geq 0,$$

$$\phi(x_0 - h) - \phi(x_0) \geq 0;$$

whence

$$\frac{\phi(x_0 + h) - \phi(x_0)}{h} \geq 0,$$

$$\frac{\phi(x_0 - h) - \phi(x_0)}{-h} \leq 0.$$

Therefore as $\phi(x)$ has a differential coefficient at x_0 the two expressions on the right side of the above inequalities tend to 0 with h . If $\phi(x_0) = 0$, i.e., $f(x_0) = C$, which proves the theorem.

(ii) *Theorem II.* If $f(x)$ is continuous and satisfies Theorem I, then $f(x)$ cannot pass from a value A to a value B without taking all the intermediate values.

The proof is similar to the proof given above of Theorem I.

THIRD LECTURE

GENERALIZATION OF THE MEAN-VALUE THEOREM

§ 18.

29. Today's lecture will deal chiefly with such results as establish the validity of the mean-value theorem

$$f(x+h) = f(x) + hf'(x+\theta h), \quad 0 < \theta < 1,$$

under conditions less restrictive than those in Chap. I of the theorem, the mean-value theorem also with less restrictive conditions may be considered generalizations of the theorem as I understand up to late textbooks on the Differential Calculus. I lay I will suggest a number of definitions from these generalizations. I present now the generalization of W. H. Young and G. H. Young, which may be enunciated as follows —

If in a given interval $a < b$ a function $f(x)$ is defined to be finite and continuous at the end points a and b (in a and b), then for every pair of points $(x, x_0 + h)$ of $a < b$ the ends being included

$$f(x_0 + h) = f(x_0) + hf'(x_0 + \theta h), \quad 0 < \theta < 1,$$

provided that at every point $x_0 + h$ the ends being excluded there is no distinction of right and left with respect to the first derivative at that point so that $D^+f(x) = D^-f(x)$ and $D^+f(x) = D_-f(x)$.

Proof:

As in Chap. I, f' is the mean-value theorem in Art. 2), let $\phi(x)$ denote

$$\phi(x) = f(x) - f(x_0) - \frac{x - x_0}{h} \{f(x_0 + h) - f(x_0)\}.$$

Then $\phi(x_0)$ and $\phi(x_0 + h)$ both vanish. As $\phi(x)$ is finite and continuous in $(x_0, x_0 + h)$ and does not show any distinction of right and left with regard to the derivative at any point inside $(x_0, x_0 + h)$, but at every such point

$$D^+\phi(x) = D^-\phi(x) \text{ and } D_+\phi(x) = D_-\phi(x).$$



Now $f(x)$ is not zero throughout $(c-\epsilon, c+\epsilon)$. It must have an upper and/or a lower bound different from zero. Also, because f has no jumps on $(c-\epsilon, c+\epsilon)$, this upper or lower bound must be attained at an interior point of $(c-\epsilon, c+\epsilon)$. Calling ξ such a point there must exist a number $\epsilon_1 > 0$ but sufficiently small so that, taking h to be always $\leq \epsilon_1$,

$$\left. \begin{aligned} \phi(\xi+h) - \phi(\xi) &\leq 0 \\ \phi(\xi-h) - \phi(\xi) &\leq 0 \end{aligned} \right\} \text{ for } h \leq \epsilon_1$$

$$\left. \begin{aligned} \phi(\xi+h) - \phi(\xi) &\geq 0 \\ \phi(\xi-h) - \phi(\xi) &\geq 0 \end{aligned} \right\} \text{ for } h \leq \epsilon_1$$

In the first alternative $D^+ \phi(\xi) = 0$ and $D^- \phi(\xi) = 0$ are both ≤ 0 and $D^+ \phi(\xi)$ and $D^- \phi(\xi)$ are both ≥ 0 . In the second alternative the above-mentioned inequalities are reversed. Therefore, $D^+ \phi(\xi) = D^- \phi(\xi) = 0$ and ϕ has a stationary function of right and left

$$\begin{aligned} D^+ \phi(\xi) &= D^- \phi(\xi) = 0 \\ D^+ \phi(\xi) &= D^- \phi(\xi) = 0, \end{aligned}$$

and no jump occurs at $x_0 = \xi$. Therefore, $D^+ \phi(\xi) = D^- \phi(\xi) = 0$ and

$$f'(\xi) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = 0,$$

$$\text{i.e., } f(x_0+h) = f(x_0) + h f'(x_0 + \theta h), \quad 0 \leq \theta \leq 1.$$

§ 10

20. A function f is said to be *piecewise monotonic* if the above generalization is true.

Def. 1. Let $a < b$. A function f is said to be *piecewise monotonic* on (a, b) if there exists a finite set $\{x_1, x_2, \dots, x_n\}$ such that $a < x_1 < x_2 < \dots < x_n < b$ and f is monotonic on each of the intervals (x_{i-1}, x_i) for $i = 1, 2, \dots, n$. The exception of f at x_i where $i = 1, 2, \dots, n$ is allowed. Although Darb's condition is not necessary for a function to be piecewise monotonic and for every pair of points $x < y$ in (a, b) it is true that

$$f(x_0+h) - f(x_0) = hf'(x_0 + \theta h), \quad 0 \leq \theta \leq 1$$

$$\begin{aligned} \text{i.e., } f(x_0+h) &= \left\{ \frac{1}{2} - \theta, \frac{1}{2} - \theta + \frac{\pi}{4} \right\} \cdot \frac{1}{2} \left(\frac{1}{2} - \theta, \frac{1}{2} + \frac{\pi}{4} \right) \\ &= h \left(\frac{1}{2} - \theta \right) \left(\frac{1}{2} + \theta \right) \quad (0 \leq \theta \leq 1) \end{aligned}$$

* Obviously, the same holds at $x = y$ for piecewise monotonic and differentiable functions.



THEOREM 1. Let $\phi(x)$ be continuous in $\frac{1}{x}$ for $1 \leq x \leq 1$ then $\phi(x)$ is continuous in the interval $x = 1$ and has a differential coefficient in $\frac{1}{x}$ $\frac{1}{x^2} \cos \frac{1}{x}$ at every point with the exception of $x = 0$ where however $\phi(x)$ is not a function of x and hence the theorem is not applicable. Therefore although the conditions are satisfied, the condition for the existence of the mean value theorem does not hold.

§ 20

The next question is to find out what conditions are necessary for the theorem to hold. —

If $\phi(x)$ is continuous in $\frac{1}{x}$ a function $\phi(x)$ is defined to be finite and continuous at a point x_0 if $\lim_{x \rightarrow x_0} \phi(x) = \phi(x_0)$, then for every pair of points x_0, x_1 in the interval (a, b) the following conditions are satisfied:

$$f(x_0 + h) = f(x_0) + h f'(x_0 + \theta h), \quad 0 \leq \theta \leq 1,$$

provided that

(i) there is no point in the interval (a, b) where $\phi(x)$ is not finite and continuous, the progressive difference quotient and the corresponding differential coefficient exists, while the other does not exist and

(ii) $\phi(x)$ is not infinite at x_0 or x_1 or any other point in the interval (a, b) and $\phi(x)$ is not infinite at x_0 or x_1 or any other point in the interval (a, b) .

Proof.

Consider the function

$$\phi(x) = f(x) - f(x_0) - \frac{1}{h} \{f(x_0 + h) - f(x_0)\}$$

Then obviously $\phi(x_0) = 0$ and $\phi(x_0 + h) = 0$ when $h = x_0 - x_0$ or $h = x_1 - x_0$ or $h = x_0 - x_1$ or $h = x_1 - x_1$. Then unless $\phi(x)$ is zero throughout the interval $x = x_0, x_1$ $\phi(x)$ has at least one maximum or one minimum inside the interval.

Suppose f has a maximum at the point ξ with $\phi(\xi) = 0$. Then, by the hypothesis (1) of the theorem, the derivatives of f satisfy either the inequalities

$$\left. \begin{aligned} D_+ \phi(\xi) &\leq 0 \\ D_- \phi(\xi) &\leq D_+ \phi(\xi) \end{aligned} \right\} \quad (A)$$

$$\left. \begin{aligned} D_+ \phi(\xi) &\geq 0 \\ D_- \phi(\xi) &\geq D_+ \phi(\xi) \end{aligned} \right\} \quad (B)$$

Suppose that the hypothesis (A) holds. Now, as ξ is a point of maximum,

$$D^+ \phi(\xi) \leq 0 \text{ and } D^- \phi(\xi) \geq 0. \quad (a)$$

But by the inequalities (A),

$$D^+ \phi(\xi) \geq D^- \phi(\xi).$$

Therefore, combining (a) with (b), we have

$$D^+ \phi(\xi) = D^- \phi(\xi) = 0. \quad (b)$$

Again, if $\phi(\xi) < 0$, for ξ is a point of maximum, but

$$D_+ \phi(\xi) \leq D^- \phi(\xi)$$

i.e., because of (b)

$$D_+ \phi(\xi) \leq 0.$$

Therefore

$$D_+ \phi(\xi) = 0 = D^- \phi(\xi)$$

Hence

$$\phi(\xi) = 0 \text{ exists and is 0 also by (b) } D^+ \phi(\xi) = 0.$$

Therefore $\phi(\xi) = 0$ exists by the hypothesis (1) of the theorem and must be 0. Thus $\phi(\xi)$ exists and is zero.

Hence

$$f'(\xi) = \frac{f(x_0 + h) - f(x_0)}{h},$$

$$\text{i.e., } f(x_0 + h) = (1 + h)f'(\xi) + f(x_0 + 0h) \quad (0 < h < 1)$$

In a similar manner, if the derivatives at ξ satisfy the inequalities (B) it can be shown that

$$f'(\xi) = D_+ \phi(\xi) = D^- \phi(\xi) = 0$$

so that $f'(\xi) = 0$ exists and is 0 also by (b) and the theorem holds.

If f is a point of minimum, it can be shown similarly that the theorem holds.

Thus the theorem is completely established.

§ 21.

The following two examples illustrate Singh's generalization.

Ex. 1. Let $f(x)$, equal to $\frac{1}{x}$ in the interval $(0, 1)$ and equal to $\frac{1}{x}$ in the interval $(-1, 0)$. Then at the point $x = 0$ inside $(-1, 1)$ there is a distinction of right and left with regard to the derivatives, for

$$D^+f(0) = 2, D_-f(0) = -2$$

$$D^+f(0) \neq 1, D_-f(0) \neq -1,$$

so that

$$D^+f(0) \neq D_-f(0) \text{ and not equal; also}$$

$$D^+f(0) \text{ is unequal to } D_-f(0).$$

At all the other points the difference is evident exists and so there is no distinction of right and left. Also $f(x)$ is not a continuous function $(-1, 1)$, the only thing needed. Still for every pair of points $(x_0, x_0 + h)$ whatever, the theorem holds.

Ex. 2. Let $f(x)$ equal to $x \log(x^2)$ in the interval $(0, 2)$ and be equal to $x \log x^2$ in the interval $(-2, 0)$. Then for

$$f(x) = \frac{\infty}{2} \frac{\psi(x-u_0)}{2^n},$$

where $\{\cdot\}$ is an enumerable and every h real $0 < h < 1$ inside $(-1, 1)$ the generalization of Singh holds, so that the continuous theorem is valid although at every point of the set $\{x_n\}$ there is a distinction of right and left with regard to the derivatives.

22

22. Singh has generalized the mean value theorem still further, by extending it to even certain types of discontinuous functions.¹ His second generalization runs as follows:

If inside a given interval $a(b)$ for which $f(x)$ is defined there is a point X at which there is a discontinuity of the second kind, at least on one side (say the right (left)), and if inside a finite interval h never small with h as point of discontinuity X as left (right) end point

¹ In his paper (c) Heine shows that the mean value theorem may hold for a discontinuous function like $\sin \frac{1}{x}$.

Now in the case of f which satisfies these conditions there always exists a number $\delta_1 > 0$ such that for any $\epsilon > 0$ and for x numerically after a_1 δ_1 and for $h = \pm \delta_1$ the ratios

$$\frac{f(a_1 + h) - f(a_1)}{h}, \quad \frac{f(a_1 - h) - f(a_1)}{h}$$

are numerically less than a finite quantity ϵ , where ϵ is numerically greater than 0, the same value of ϵ being inferred in absolute value

of $\frac{f(x) - f(a_1)}{x - a_1}$ on the maximum value of $|f(x) - f(a_1)|$ in the given interval, where $|f(x) - f(a_1)| < \epsilon$ (the value of the quantity $\frac{f(x) - f(a_1)}{x - a_1}$ and

suppose that ϵ is chosen so that the interval from $a_1 - \epsilon$ to $a_1 + \epsilon$ is at least one of the points $a_1 - \epsilon, a_1, a_1 + \epsilon$, the point a_1 , in such a manner that we have

$$x = a_1 \mp \delta, \quad x + h = a_1 \pm \delta'$$

where δ and δ' are positive, we shall have in absolute value

$$f(a_1 \pm \delta') - f(a_1) < \delta' A', \quad f(a_1 \mp \delta) - f(a_1) < \delta A',$$

and hence also in absolute value

$$f(a_1 \pm \delta') - f(a_1 \mp \delta) < (\delta + \delta') A',$$

and consequently

$$f(x + h) - f(x) = h h_1 A'.$$

hence h_1 is given and $y = f(x)$ is a function of x , and it is evident that f is continuous.

Now this w may be regarded as a generalization in appearance of the law given by Hurwicz and Thomas¹ independent of such a law. It may be stated as follows: If in an interval (a, b) $f(x)$ is defined as a continuous function and if one of the four derivatives, say the upper derivative in the right, is a continuous function of x on the interval, then for any two points $x, x + h$ in (a, b) $f(x + h) - f(x) = h f_+(x, h)$ for $0 < h < 1$.

The continuity of the upper derivative carries with it the existence of the differential coefficient and is equivalent with the derivative. So really the theorem is a generalization.

¹ Hurwicz, *Die Formen der Funktion*, Leipzig, 1910; Thomas, *Existenz in der Theorie der Besten*, Leipzig, 1911, p. 1.

§ 24.

The following two theorems are worthy of mention as generalizations of Rolle's theorem.

(a) *Mean Value Theorem*. If $f(x)$ is a function of x which is finite and continuous in an interval (a, b) , the ends being included, and is also differentiable at every point ξ inside the interval (a, b) , then there exists at least one of the upper derivatives at a point ξ inside the interval (a, b) which is equal to the slope of the secant line, that is

$$D^+ f(\xi) = \frac{f(b) - f(a)}{b - a}.$$

or the converse, namely, if $f(x)$ is differentiable at ξ and right

$$D^+ f(\xi) = \frac{f(b) - f(a)}{b - a},$$

Proof

(1) If $f(x)$ is constant the theorem is evident and the derivatives being zero everywhere. If not, $f(x)$ has a positive upper bound or a negative lower bound, and being a continuous function there is a point ξ inside (a, b) where $f(x)$ assumes such an extreme value.

It then follows as in Art. 23 that

$$f(x) - f(a) = \int_a^x f'(\xi) d\xi$$

are respectively ≤ 0 and ≥ 0 if $f(x)$ is a maximum, the signs in the integrand are respectively ≤ 0 and ≥ 0 if $f(x)$ is a minimum.

Therefore, $f'(x) = 0$ at the point ξ where the derivatives on one side of ξ are ≤ 0 and on the other ≥ 0 which proves the theorem.

(2) *Mean Value Theorem*. If $f(x)$ is a finite and continuous function defined in an interval (a, b) , the ends being included, and is such that the upper and lower derivatives on one side of ξ are equal to the upper and lower derivatives on the other side, then there is a point ξ inside the interval (a, b) where the upper and lower derivatives of the two

$$\begin{aligned} D^+ f(\xi) &= \frac{f(b) - f(a)}{b - a} \\ D^- f(\xi) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

exist and are equal.

By the Mean Value Theorem, $f(b) - f(a) = f'(\xi)(b - a)$ for some ξ in the interval (a, b) . The point ξ is such that the upper and lower derivatives at ξ are equal.

87. I will consider the case $\lim_{x \rightarrow x_0} f'(x) = G$ for x increasing from x_0 on the right and $x = x_0$ being a point of continuity of f .

Theorem 1¹

If y be a continuous function of x in the interval (x_0, x_1) and f has upper derivative to the right G everywhere G then the limit to the right ($G > 0$), then the quotient

$$\frac{y' - y}{x' - x}$$

also lies between the same limits whatever arbitrary variable taken for x and x' in (x_0, x_1) .

Proof.

Let $x' > x$. If $\frac{y' - y}{x' - x}$ were $> G$, then a quantity $\epsilon > 0$ could be chosen to be so small that, if x were defined by

$$x = y - (G + \epsilon)x$$

the difference $x' - x$ would be > 0 , say equal to δ . Therefore there must exist for x an upper bound x_1 ($x_1 < x$) where the relation

$$x' - x'' = \delta$$

would be satisfied. At the point x_1 we find

$$D^+ y \geq G + \epsilon$$

and, consequently,

$$D^+ y' \geq G + \epsilon,$$

which would be against the hypothesis.

If, however, $\frac{y' - y}{x' - x}$ were $< G$, then a quantity $\epsilon > 0$ could be chosen to be so small that, if x were defined by

$$x = y - (G - \epsilon)x$$

the difference $x' - x$ would be < 0 , say equal to $-\delta$. Therefore there must exist for x an upper bound x_1 ($x_1 < x$) where the relation

$$x' - x'' = -\delta$$

would be satisfied. At the point x_1 we find

$$D^+ y \leq G - \epsilon$$

or

$$D^+ y' \leq G - \epsilon$$

what is again against the hypothesis.

If in the theorem instead of the upper derivative on the right we take any of the other derivatives the theorem will remain true and the proof will be similar to that given above.

¹ The enunciation as well as the proof is almost word for word the same as in Sebestien's paper in *Acta Math.* Vol. 5 (1911) p. 19.



Theorem I. If a continuous function has the four derivatives and have the same upper bound and the same lower bound in a given interval and the same upper and lower bounds for the upper and lower bounds of the difference quotient

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

where x_1, x_2 are any possible values in a given interval.

Proof.

Let U and L denote respectively the upper bound and the lower bound of

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then it is obvious from Theorem I that

$$U \leq 0, L \leq 0'.$$

We proceed to show that in the above the only signs permitted are those of equality.

For if possible suppose for example that $U < 0$ and is equal say to $0 - \eta$ where $\eta > 0$. Thus it is not possible to find two values x_1, x_2 which will give

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 - \eta.$$

But this is absurd because from the fact that the upper bound of $D f(x)$ is $0 - \eta$ it follows that however small a positive number ϵ may be chosen there are values x_1 and h such that

$$\frac{f(x_1 + h) - f(x_1)}{h} > 0 - \epsilon.$$

Similarly it can be proved that L cannot be < 0 . Thus it is proved that the upper derivative and the incremental ratio have the same upper and lower bounds.

In the same manner, it can be proved that any of the other derivatives has the same upper and lower bounds as the incremental ratio.

Theorem II. If one of the derivatives of a continuous function has a positive lower bound then the function is monotone and increasing; if one of the derivatives of a continuous function has a negative upper bound then the function is monotone and decreasing.

This follows immediately from Theorem J.

Theorem I. If a continuous function $f(x)$ has a derivative which is continuous at a point, then $f(x)$ has a unique derivative at that point and $f(x)$ has a differential coefficient at that point.

Proof:

Take any interval $(x_0 - \delta, x_0 + \delta)$ from the δ -neighbourhood of x_0 . In this interval the upper and lower limits of the difference $D_\delta f(x)$ are respectively the upper and lower limits of an ϵ -derivation of $f(x)$ at x_0 . As $D_\delta f(x)$ is continuous at x_0 , the upper and lower limits of $D_\delta f(x)$ at x_0 are $D_\delta f(x_0)$, each of these from $D_\delta f(x)$ by a theorem which holds for any function and, therefore, for a derivation. Hence we may use for the other three derivations $D_\delta f(x_0)$ as the ϵ -derivations. As $D_\delta f(x_0)$ is continuous at x_0 , hence the ϵ -derivations $D_\delta f(x_0)$ are the same for all δ sufficiently small.

Let $M = M_\delta$. A function $f(x)$ is said to have a unique derivative constant if we know $\lim_{x \rightarrow x_0} D_\delta f(x) = L$ for all values of the variable.

Proof 1:

Let $\phi(x) = f(x) - Lx$. Then $\phi(x)$ has the same derivative as $f(x)$ and the upper derivative in the interval $(x_0 - \delta, x_0 + \delta)$ is $\phi(x)$.

$$\phi(x) \leq cx + b(x) \leq f(x),$$

where $c > 0$ is an arbitrary small quantity.

(*) It can be proved as follows that

$$b(x) \leq 0.$$

If c is a constant, then $\phi(x) \leq cx + b(x)$ for every x and c is arbitrary. Thus $b(x) \leq 0$ for every x and $b(x) \leq 0$ for every x .

$$\frac{f(x+h) - f(x)}{h} \geq D^+ f(x) - \epsilon$$

Also for a fixed x and ϵ we can choose h sufficiently small that

$$\frac{f(x+h) - f(x)}{h} < D^+ f(x) + \epsilon.$$

Therefore it follows that for a fixed x and ϵ there exist h such that

$$\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h} > -\epsilon.$$

$$\text{i.e.} \quad \frac{\phi(x+h) - \phi(x)}{h} > -\epsilon - 2\epsilon.$$

If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary.

$$D^+\phi(x) \geq c$$

If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary. If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary. If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary.

$$\phi(x'') - \phi(x') = 0.$$

and for every value of $h < x' - x''$ the relation

$$\phi(x'' + h) - \phi(x'') < 0$$

holds, which would be a contradiction with the assumption that $D^+\phi(x'') \geq c$.

Hence $\phi(x'') = \phi(x'')$ for all x'' in I . If $\phi(x'') < \phi(x')$ for some x'' in I , then $\phi(x'') - \phi(x') < 0$ which is impossible.

Interchanging D^+ and D^- in the above, it follows that

$$\phi(x'') = \phi(x') \text{ for all } x'' \text{ in } I. \text{ If } \phi(x'') > \phi(x') \text{ for some } x'' \text{ in } I, \text{ then } \phi(x'') - \phi(x') > 0 \text{ which is impossible.}$$

Hence $\phi(x'') = \phi(x')$ for all x'' in I . If $\phi(x'') > \phi(x')$ for some x'' in I , then $\phi(x'') - \phi(x') > 0$ which is impossible.

Hence it must be equal to 0.

Theorem V. If $\phi(x)$ is a function of x which is continuous on I and $\phi'(x)$ exists at every point in I , then $\phi(x)$ is a continuous function.

Theorem VI. If $\phi(x)$ is a function of x which is continuous on I and $\phi'(x)$ exists at every point in I , then $\phi(x)$ is a continuous function.

Theorem VII. If $\phi(x)$ is a function of x which is continuous on I and $\phi'(x)$ exists at every point in I , then $\phi(x)$ is a continuous function.

Theorem VIII. If $\phi(x)$ is a function of x which is continuous on I and $\phi'(x)$ exists at every point in I , then $\phi(x)$ is a continuous function.

$$D^+\phi(x)$$

If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary. If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary. If $\phi(x) \leq \phi(x') \leq \phi(x'') \leq \phi(x''')$ for all x, x', x'', x''' in I , then ϕ is arbitrary.

$$L < K < U,$$

Now, equate the coefficient of h^{n+1} on both sides of (1) to get

$$f^{(n+2)}(x) = \sum_{m=1}^{\infty} f^{(n+1)}(x) \cdot T_{n+1,m} \quad n \geq 0 \quad (2)$$

where $T_{n+1,m}$ denotes the coefficient of h^m in the expansion of

$$\left\{ \sum_{r=0}^{\infty} A_r h^r \right\}^K.$$

From (2) we get the following equations first given by Whittem² —

$$\frac{1}{2!} f^{(2)}(x) = A_0 f^{(0)}(x),$$

$$\frac{1}{3!} f^{(3)}(x) = A_1 f^{(2)}(x) + \frac{A_0^2}{2!} f^{(0)}(x)$$

$$\frac{1}{4!} f^{(4)}(x) = A_2 f^{(2)}(x) + \frac{2A_0 A_1}{2!} f^{(0)}(x) + \frac{A_0^3}{3!} f^{(0)}(x)$$

$$\begin{aligned} \frac{1}{5!} f^{(5)}(x) = & A_3 f^{(2)}(x) + \frac{2A_0 A_2}{2!} f^{(0)}(x) + \frac{A_1^2}{2!} f^{(0)}(x) + \frac{3A_0^2 A_1}{3!} f^{(0)}(x) \\ & + \frac{A_0^4}{4!} f^{(0)}(x), \end{aligned}$$

$$\begin{aligned} \frac{1}{6!} f^{(6)}(x) = & A_4 f^{(2)}(x) + \frac{2A_0 A_3}{2!} f^{(0)}(x) + \frac{A_1 A_2}{2!} f^{(0)}(x) + \frac{3A_0^2 A_2}{3!} f^{(0)}(x) \\ & + \frac{3A_0 A_1^2}{3!} f^{(0)}(x) + \frac{6A_0^3 A_1}{4!} f^{(0)}(x) + \frac{A_0^5}{5!} f^{(0)}(x) \end{aligned}$$

$$\begin{aligned} \frac{1}{7!} f^{(7)}(x) = & A_5 f^{(2)}(x) + \frac{2A_0 A_4}{2!} f^{(0)}(x) + \frac{2A_1 A_3}{2!} f^{(0)}(x) + \frac{A_2^2}{2!} f^{(0)}(x) \\ & + \frac{4A_0^2 A_3}{3!} f^{(0)}(x) + \frac{A_1^2 A_2}{3!} f^{(0)}(x) + \frac{6A_0 A_1 A_2}{3!} f^{(0)}(x) \\ & + \frac{A_0^4 A_1}{4!} f^{(0)}(x) + \frac{6A_0^3 A_2}{5!} f^{(0)}(x) \end{aligned}$$

* For large values of N the expression for $T_{N,K}$ is rather complicated. See recurrence formulae for the calculation in a later paper in *J. M. S.* Vol. 14, 1975, pp. 79-84; also see B. Hattestad's paper in *Teorijsko Matematiški Vestnik* Vol. 2, 1963, pp. 13-16.

² *Id.* p. 36.

$$+ \frac{A_1}{4} A_2 f^{(1)}(x) + \frac{A_2^2}{4} f^{(2)}(x) + \frac{A_1^3}{4} f^{(3)}(x) \\ + \frac{A_1^2}{4} f^{(4)}(x)$$

$$\frac{1}{6} f^{(3)}(x) = \frac{A_1^3}{6} f^{(3)}(x) + \frac{A_1^2 A_2}{6} f^{(4)}(x) + \frac{A_1 A_2^2}{6} f^{(5)}(x) + \frac{A_2^3}{6} f^{(6)}(x)$$

$$\frac{1}{6} f^{(3)}(x) = \frac{A_1^3}{6} f^{(3)}(x) + \frac{A_1^2 A_2}{6} f^{(4)}(x) + \frac{A_1 A_2^2}{6} f^{(5)}(x) + \frac{A_2^3}{6} f^{(6)}(x)$$

$$\frac{1}{6} f^{(3)}(x) = \frac{A_1^3}{6} f^{(3)}(x) + \frac{A_1^2 A_2}{6} f^{(4)}(x) + \frac{A_1 A_2^2}{6} f^{(5)}(x) + \frac{A_2^3}{6} f^{(6)}(x)$$

$$\frac{1}{6} f^{(3)}(x) = \frac{A_1^3}{6} f^{(3)}(x) + \frac{A_1^2 A_2}{6} f^{(4)}(x) + \frac{A_1 A_2^2}{6} f^{(5)}(x) + \frac{A_2^3}{6} f^{(6)}(x)$$

$$\diamond \frac{6A_2^3}{6!} f^{(6)}(x) + \frac{A_2^2}{7!} f^{(7)}(x).$$

Two more equations have been given by (1) and (2), those corresponding to

$$\frac{1}{6!} f^{(6)}(x) \quad \text{and} \quad \frac{1}{7!} f^{(7)}(x).$$

From the first equation we get A_2 using the value of A_1 in the second we get A_3 and so on.

The computed values of the first eight A 's are

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{1}{24} f^{(1)}(x), \quad A_2 = \frac{1}{2} \left[\frac{1}{24} f^{(2)}(x) - \frac{1}{24} \left\{ \frac{f^{(1)}(x)}{2} \right\}^2 \right]$$

$$A_3 = \frac{1}{6} \left[\frac{11}{24} f^{(3)}(x) - \frac{f^{(1)}(x) f^{(2)}(x)}{2} + \frac{11}{24} \left\{ \frac{f^{(1)}(x)}{2} \right\}^3 \right]$$

$$A_4 = \frac{1}{24} \left[\frac{11}{240} f^{(4)}(x) - \frac{11}{120} \left\{ \frac{f^{(1)}(x)}{2} \right\}^2 f^{(2)}(x) + \frac{7}{2} \left\{ \frac{f^{(1)}(x)}{2} \right\}^2 f^{(3)}(x) - \frac{1}{10} \left\{ \frac{f^{(1)}(x)}{2} \right\}^4 \right]$$



$$A_2 = \frac{1}{5!} \left[\frac{1}{400} \frac{f}{f^2} - \frac{1}{300} \frac{f^2}{(f^2)^2} + \frac{1}{24} \frac{f^3}{(f^2)^3} - \frac{1}{120} \frac{\{f^4\}}{\{f^2\}^2} \right. \\ \left. + \frac{1}{10} \frac{\{f^3\}^2}{(f^2)^4} - \frac{1}{20} \frac{f^4}{(f^2)^2} + \frac{1}{11} \frac{f^5}{(f^2)^5} \right].$$

$$A_3 = \frac{1}{6!} \left[\frac{15}{800} \frac{f^5}{f^2} - \frac{1}{16} \frac{f^6}{(f^2)^2} + \frac{1}{16} \frac{f^7}{(f^2)^3} - \frac{1}{7} \frac{f^8}{(f^2)^4} \right. \\ \left. + \frac{11}{112} \frac{\{f^3\}^2 f^2}{(f^2)^5} - \frac{2}{192} \frac{\{f^2\}^2 \{f^3\}^2}{(f^2)^6} + \frac{1}{24} \frac{f^9}{(f^2)^5} \right. \\ \left. + \frac{17}{14} \frac{f^4 f^5}{(f^2)^5} - \frac{11}{128} \frac{\{f^3\}^2}{(f^2)^4} + \frac{1}{64} \frac{f^4 f^5}{(f^2)^2} + \frac{8}{121} \left\{ \frac{f^6}{f^2} \right\} \right]$$

$$A_4 = \frac{1}{7!} \left[\frac{237}{16128} \frac{f}{f^2} - \frac{17}{224} \frac{f^2}{(f^2)^2} + \frac{1}{8064} \frac{f^3}{(f^2)^3} - \frac{1}{9216} \frac{f^4}{(f^2)^4} \right. \\ \left. + \frac{\{f^3\}^2 f^2}{(f^2)^5} + \frac{1}{2048} \frac{\{f^2\}^2 \{f^3\}^2}{(f^2)^6} - \frac{90}{2048} \frac{\{f^3\}^2 f^3}{(f^2)^5} \right. \\ \left. + \frac{135}{224} \frac{\{f^3\}^2 f^4}{(f^2)^6} - \frac{21}{128} \frac{f^5}{(f^2)^4} - \frac{287}{128} \frac{f^6}{(f^2)^5} \right. \\ \left. - \frac{8243}{65536} \frac{\{f^3\}^2 f^5}{(f^2)^6} + \frac{111}{9216} \frac{f^7}{(f^2)^4} + \frac{1}{4} \frac{f^8}{(f^2)^5} \right. \\ \left. + \frac{105}{64} \frac{f^5 f^6}{(f^2)^6} - \frac{71}{512} \frac{f^4 f^7}{(f^2)^5} - \frac{11}{128} \left\{ \frac{f^7}{f^2} \right\} \right]$$

(2)

79. The next problem of Mr. Whitcomb goes back according to Professor R. R. R. to the time of Cauchy and may be enunciated as follows. Prove that $m = \frac{1}{h} \ln \frac{h}{h-2}$.

we have

$$\frac{h^2}{2} f''(x) = \frac{h}{6} [f'(x + h) + f'(x - h)] - h^2 \theta_2, \quad \text{where } \theta_2 \text{ tends to } 0 \text{ with } h. \quad (3)$$

Now from (2) $\xi = h\theta$, $h\theta \rightarrow 0$, where θ tends to 0 with h . Therefore (3) becomes, on dividing by h^2 ,

$$\frac{1}{2} f''(x) = \frac{1}{6} [f'(x + \theta_1 h) + f'(x - \theta_2 h)] - \theta_2, \quad \text{where } \theta_1, \theta_2 \text{ tend to } 0 \text{ with } h.$$

$$f'(x + \theta_1 h) = \frac{h}{24} f''''(x) + \frac{h}{6} \left\{ \frac{1}{2} f''(x + \theta_1 h) + \frac{1}{2} f''(x - \theta_2 h) \right\} + \frac{h}{24} f''''(x) + \frac{h}{6} f''(x) \quad (4)$$

putting

$$f'(x + \theta_1 h) = f'(x) + \theta_1 f''(x) + \theta_1^2 f'''(x) + \dots \quad \text{where } \theta_1 \text{ tends to } 0 \text{ with } h$$

Now $f'''(x) \neq 0$, therefore dividing both the sides of (4) by $f''(x)$

$$a \approx \frac{h}{24} \frac{f''''(x)}{f''(x)} + E$$

$$\text{where } E = \frac{h}{24} \left\{ \frac{1}{2} \left(\frac{f''''(x + \theta_1 h)}{f''(x + \theta_1 h)} + \frac{f''''(x - \theta_2 h)}{f''(x - \theta_2 h)} \right) + \frac{1}{2} \left(\frac{f''''(x)}{f''(x)} + \frac{f''''(x)}{f''(x)} \right) \right\}$$

Therefore E may be neglected in comparison with

$$\frac{h}{24} \frac{f''''(x)}{f''(x)}$$

as $f''(x)$ is finite by hypothesis

Thus it is proved that

$$\lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{h} \right) f''(x) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{h} \right) f''(x) = \frac{1}{24} \frac{f''''(x)}{f''(x)}$$

42. If $f''(x)$ exists and is finite with all its regard to x and may be the behaviour of $f''''(x)$ at points other than x , the above result holds, it being assumed that in addition to the function $\theta(h)$ being single-valued

$f(x)$, $f'(x)$, $f''(x)$, ... are continuous on an interval $x < x < x$. The formula for the Taylor expansion of $f(x)$ about x is then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}\{f^{(n)}(x) + \epsilon\}.$$

The error term ϵ is a function of x and h which is different from zero only when $h \neq 0$. It is a function of x and h which is different from zero only when $h \neq 0$.

2nd

The first part of the proof of the mean value theorem is given in the next lecture. The second part of the proof is given in the next lecture.

Proof of the first part of the proof of the mean value theorem. Let $f(x)$ be a function of x which is continuous on the interval $x < x < x$. For the purpose of the proof, let us assume that $f(x)$ is a function of x which is continuous on the interval $x < x < x$. Taking for the sake of simplicity

$$x=0, f(h) = \int_0^h \phi(t) dt, f' = \phi(h)$$

in the mean value theorem

$$f(x+h) = f(x) + hf'(x+\theta h) \quad \text{---}$$

The proof. If $f(x)$ is a function and $f'(x)$ is a function of x then there is a unique function $f(x)$ between h and h varying in the domain Δ and $f'(x)$ is a function of x .

The proof. If $f(x)$ is a function of x then the function $f(x)$ must not have an absolute maximum or minimum in the neighborhood of any point x in the domain Δ .

Example of the proof. Let $f(x) = x^2$. Also see the proof of the mean value theorem in the next lecture.

Proof of the second part of the proof of the mean value theorem. Let $f(x)$ be a function of x which is continuous on the interval $x < x < x$. For the purpose of the proof, let us assume that $f(x)$ is a function of x which is continuous on the interval $x < x < x$. Taking for the sake of simplicity

$$x=0, f(h) = \int_0^h \phi(t) dt, f' = \phi(h)$$

in the

* See pp. 100-101 of the next lecture.

Cor. As $n \rightarrow \infty$, $h \rightarrow 0$, then θ has an infinite number of maxima and minima in the neighborhood of every point. It follows from the above theorem that if θ is a single-valued function, it cannot be a nowhere differentiable function.

44. As in his first paper, *He* had postulated that θ should be not only single-valued but also differentiable. Prasad pointed out in the following theorems that the single-valuedness of θ carried with it as a consequence its continuity but not its differentiability.

Theorem S. If θ is a single-valued function of h , then it is necessarily continuous everywhere with the possible exception of $h = 0$.

Theorem T. If θ is a single-valued and continuous, $\theta'(h)$ need not exist for every value of h .

45. Prasad gives certain general types* of $\theta(h)$ with the necessary and sufficient conditions for the existence of $\theta(+0)$ for each type. The types are the following:

$$\text{Type I. } \theta(t) = \int_0^t (2 + \sin \sqrt{v}) dv, \quad 0 \leq t < \infty.$$

Here $\theta(+0)$ exists or does not exist according as

$$\psi \approx \log \frac{1}{v} \quad \text{or}$$

$$\psi \approx \log \frac{1}{v}.$$

Proof.

(i) Let $\psi \approx \log \frac{1}{v}$. Then $\psi(0)$ exists and equals $-\frac{1}{2}$, that is $\theta(+0)$ exists and is $-\frac{1}{2}$. Hence Dirichlet's condition is satisfied at $h = 0$ as the integrand being always positive, $\theta(t)$ is monotonic. Then θ is single-valued and $\theta(+0) = \frac{1}{2}$.

(ii) Let $\psi \approx \log \frac{1}{v}$. Then it can be seen without difficulty that $\theta(t) = 2t + \Lambda t \cos \{\sqrt{t} + 1\} + O(1)$ where Λ is a constant different from zero and B is another constant.

* See I, c. p. 3-4.

* See pp. 172-74 of his first paper. I take this opportunity to point out a error in that paper at the end of p. 173. The condition should be $\psi \approx \log \frac{1}{v}$ or $\psi \approx \log \frac{1}{v}$.

* See Prasad's paper "On the differentiability of the integral function" *Ceylon Journal of Science* (1929) also Prasad's papers in the *Bulletin of the Cal. M. S.* Vol. II.

Also $f(h) = h^2 + A_1 h^2 \cos \{ \sqrt{h} + \frac{1}{2} \} + A_2$ and a constant $\neq 0$ and B_1 another constant.

Now by the mean value theorem *

$$f(h) = h \omega(\theta h),$$

Therefore

$$\theta(h \rightarrow 0) = \text{non-existent as } \lim_{h \rightarrow 0} \frac{1 + A_1 \cos \{ \sqrt{h} + \frac{1}{2} \}}{2 + A_2 \cos \{ \sqrt{h} + \frac{1}{2} \}} \text{ is non-existent,}$$

because if the limit $\theta(h \rightarrow 0)$ existed

$$\lim_{h \rightarrow 0} \frac{1 + A_1 \cos \{ \sqrt{h} + \frac{1}{2} \}}{2 + A_2 \cos \{ \sqrt{h} + \frac{1}{2} \}} = \theta \quad \dots \quad (1)$$

would exist which is not possible as $h \rightarrow 0$ cannot be 0 as is obvious from the form of the expression (1)

Type II $\phi(t) = \int_0^t \psi(r) (2 + \sin \sqrt{r} + r) dr$ $\lambda \sim 1$ $\theta(h \rightarrow 0)$ exists or does not exist according to

$$\psi(r) = \frac{1}{\log \frac{1}{r}},$$

$$\psi \leq \log \frac{1}{r},$$

Proof.

(a) Let $\lambda = 1$. Then denoting $\int_0^t \psi(r) dr$ by $X_1(t)$,

$$\phi(t) = 2X_1(t) + \frac{1}{2} t^2 \sin \sqrt{t} + \frac{1}{2} t^2 \quad (1)$$

$$\phi(h) = 2X_2(h) + \frac{1}{2} h^2 \sin \sqrt{h} + \frac{1}{2} h^2 \quad (2)$$

where $X_2(h)$ denotes $\int_0^h \psi_1(t) dt$

But by the mean value theorem

$$\phi(h) = h \omega(h)$$

* These are the correct conditions. In the first paper of Eves and Eves pages 2 and 3 obviously violate the conditions given. Art. 14 and satisfy the correct conditions. The error seems to have crept in by some inadvertence.

Then, dividing both sides of the above equation by $2X_2(h)$ and using (1) and (2), we have

$$1 - \frac{h}{2X_2(h)} \{ \psi'(h) \}^2 = \sin \psi(h) \frac{X_2(h^2)}{2X_2(h)} = \frac{X_1(h)}{X_2(h)} = \frac{h_1(h)}{2X_2(h)} \sqrt{h} \\ \approx \cos \psi(h) + \frac{h_1(h)}{2X_2(h)} \quad (3)$$

Now let h tend to 0; then (3) gives

$$1 = \lim_{h \rightarrow 0} \left\{ \frac{hX_1(h)}{2X_2(h)} \right\};$$

the other terms in the equation being ψ tending to 0. But X_2 is of the same order as hX_1 ; therefore the right-hand side gives a function of θ say $\phi(\theta)$, whence we get $\theta(+0)$.

b. The other case may be dealt with by using the methods of Arts. 10-11 of PRINGS' paper on *On the Function* (190), and it will be found that $\theta(+0)$ is non-existent.

46. Illustrative Examples.

Ex. 1. Let $u(t) = \int_0^t \left\{ 2 + \sin \frac{1}{\sqrt{v}} \right\} dv$

Then $\theta(+0)$ exists and is equal to $\frac{1}{2}$, although

$$f'(t) = \lim_{h \rightarrow 0} \frac{u(h)}{h} = \infty \quad \text{and} \quad f(h) = h^{\frac{1}{2}} \left(2 + \sin \frac{1}{\sqrt{h}} \right)$$

Ex. 2. Let $u(t) = \int_0^t v^{\frac{1}{2}} \left(2 + \sin \frac{1}{\sqrt{v}} \right) dv$

Then $f'(0) = \infty$ and still $\theta(+0)$ exists and is $\frac{1}{2}$.

Ex. 3. Let $u(t) = \int_0^t \left(2 + \sin \log \frac{1}{v} \right) dv$

Then $\theta(+0)$ is non-existent.

§ 29

(7) For all $\epsilon > 0$ given, choose δ such that $|f(x) - f(y)| < \epsilon$ with the δ as any in Art. 28. Then δ is a function of ϵ and ψ . For each type the types are the following:

Type I: $\psi = 1$. $\forall \epsilon > 0$, $\delta = \epsilon$. Type II: $\psi = \frac{1}{2}$ from Art. 15 that $\delta(\epsilon + 0) = \frac{1}{2}$. Now, as $\psi > \log \frac{1}{\epsilon}$, $\delta = \frac{1}{2}$.

$$f(x) = 2f\left(\frac{x}{2}\right) + \int_{x/2}^x f(t) dt - \int_{x/2}^x f(t) dt = f(x) \quad (1)$$

$$h(x) = \int_{x/2}^x f(t) dt - \int_{x/2}^x f(t) dt = \frac{f(x) - f(x/2)}{2} = \frac{f(x) - f(x/2)}{2}$$

But by the mean value theorem $f(x) - f(x/2) = h(x) - h(x/2)$.

Therefore

$$h(x) = \frac{f(x) - f(x/2)}{2} = \frac{1}{2} \left\{ f(x) - f\left(\frac{x}{2}\right) \right\} = \frac{1}{2} h(x) \quad (2)$$

Dividing (1) by the same (2) above by $h(x)$

$$1 = \frac{f(x) - f(x/2)}{h(x)} = \frac{1}{2} \left\{ \frac{f(x) - f(x/2)}{h(x) - h(x/2)} \right\} = \frac{1}{2} \cdot 2$$

Now

$$\lim_{h \rightarrow 0} \frac{h(x) - h(x/2)}{h(x)} = 1 \quad \text{as } \frac{h(x) - h(x/2)}{h(x)} \rightarrow 1 \quad \text{as } h \rightarrow 0$$

Therefore $\delta(\epsilon + 0) = \frac{1}{2}$ and $\delta(\epsilon + 0) = \frac{1}{2}$ according as $\psi > \frac{1}{2}$ or not when $\epsilon \rightarrow 0$ exists or not.

Summing up regarding Type II $\psi = \frac{1}{2}$ from Art. 15 that if ψ or $\frac{1}{2}$ exists or not according as $\frac{f(x) - f(x/2)}{h(x) - h(x/2)} \rightarrow 1$ or not.

$$\text{Type III: } \psi = \frac{1}{2} = \int_{x/2}^x f(t) dt - \int_{x/2}^x f(t) dt = 2 + \int_{x/2}^x f(t) dt$$

$$\psi(x) = 2 + \phi(x) \quad \text{as } \psi > 1$$

See page 114-17 of his first paper.



It has been proved by Prasad that $\theta'(0)$ exists or not according as

$$\psi \sim \log \frac{1}{u}$$

or

$$\psi \sim \log \frac{1}{u}$$

$$\text{Type IV} \quad x(t) = \int_0^t W(u) du, \quad W(u) = 2 + \int_0^u Y(u) du,$$

$$Y(u) = \chi(u) \{2 + \sin \phi(u)\}, \quad \chi \sim 1, \quad \phi \sim 1$$

$\theta'(0)$ exists or not according as

$$\psi \sim \log \frac{1}{u}$$

or

$$\psi \sim \log \frac{1}{u}$$

48. Illustrative Examples

$$\text{Ex. 1} \quad \text{Let } x(t) = \int_0^t W(u) du, \quad W(u) = 2 + \int_0^u Y(u) du,$$

$$Y(u) = u^{-1} \left\{ 2 + \sin \frac{1}{\sqrt{u}} \right\}$$

Then $\theta'(0)$ exists and is ∞ although $f'(0) = \infty$ and

$$f'(h) = h^{-1/2} \left\{ 2 + \sin \frac{1}{\sqrt{h}} \right\}.$$

$$\text{Ex. 2} \quad \text{Let } x(t) = \int_0^t \left\{ 2 + \sin \frac{1}{\sqrt{u}} \right\} du$$

Then $\theta'(0)$ exists and equals $\frac{1}{2}$ but $\theta''(0)$ is not existent

§ 30.

49. I will conclude today's lecture by going back to Prasad's work. Functions θ each single valued and each non-differentiable at the points of an everywhere dense set.²

² See pp. 175-80 of Prasad's 5th paper.

Let f be a function defined on $[a, b]$ such that $f'(x) = g(x)$. Then f is

increasing if $g(x) \geq 0$ for all $x \in [a, b]$, and decreasing if $g(x) \leq 0$ for all $x \in [a, b]$.

Proof. Suppose $g(x) \geq 0$ for every x in $[a, b]$. Then for every x in $[a, b]$, $f'(x) \geq 0$. If $x < y$, then by the Mean-Value Theorem, there is a c in (x, y) such that

$$f(y) - f(x) = f'(c)(y - x) \geq 0.$$

$$f(y) \geq f(x) \text{ for all } x < y \text{ in } [a, b].$$

Therefore f is increasing. Similarly, if $g(x) \leq 0$ for every x in $[a, b]$, then $f'(x) \leq 0$ for every x in $[a, b]$, and f is decreasing.

(b) Let $\{x_n\}$ be the set of rational points in the interval $(0, 1)$.

$$f(h) = \sum_{n=1}^{\infty} \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

will give $f(h) = 0$ and $f(h) = \sum_{n=1}^{\infty} \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ which is 0 for all h in $(0, 1)$.

increasing and $f(h) = 0$ for every value of h in $(0, 1)$ because each x_n is a rational number.

Therefore by Theorem 4, there is no one-to-one correspondence between h and $f(h)$ in each interval of rationality. If h is a rational number, then there is a corresponding value of f , say $f(h)$. Now, by the mean value theorem

$$f(h) = f(h_0) + f'(c)(h - h_0)$$

But $f(h)$ exists for every value of h .

$$f(h) = \left[\frac{d}{dx} f(x) \right]_{x=h_0}^{x=h} = \left\{ \frac{d}{dx} f(x) \right\}_{h=h_0}$$

Thus it is proved that $f(h) = f(h_0) + f'(c)(h - h_0)$.

$$\frac{d}{dx} f(x) \text{ exists and } f'(c) = \frac{f(h) - f(h_0)}{h - h_0}$$



But $\left(\frac{d\xi}{dh}\right)_{h=h_m}$ must not exist, for if it were to exist $\omega'(\xi_m)$ would

exist and $\theta(\xi_m) = q_m \left[\frac{d}{dh} \{u_m(h)\} \right]_{h=h_m}$, which will be absurd as $\left(\frac{d}{dh}\right)_{h=h_m}$

$\omega'(\xi_m)$ is non-existent.

If θ is included in $\{\omega_m\}$ it can be proved that $\theta \neq 0$ is non-existent because of the inclusion of the term $h^2 \rightarrow \lim_{h \rightarrow 0} \frac{1}{h^2} \theta f_{1/2}$.

Set $f = f(h)$ as a function taken for $h \in H$. Type B—The three non-differentiable functions given by Broden in *Exposition Journal*, Vol. 118 are

all non-differentiable and continuous. Therefore if $f(h) = \int_0^h u(t) dt$

where $u(t)$ is one of such functions then will be one to one correspondence between h and ξ as each moves in its domain of variability. Now each of the functions is non-differentiable at the points of an everywhere dense set, enumerable or unenumerable. Thus there are, corresponding to these functions of Broden, functions θ such of which ω is differentiable at an everywhere dense set, enumerable or unenumerable.

Denoting in general the function by $f(x)$ for $x \in H$ where H is a set of points Broden gives the following about the three functions: (1) The function is continuous and throughout coincides with x the derivative $f'(x)$ and $f''(x)$ are not well defined (finite and infinite) at two 0-sets are assigned to one point (and to $f'(x)$) with the exception of an enumerable and everywhere dense set of x -values (11). These correspond to the primary points. (See p. 27 and 28 of Broden's paper.) (2) For a function x is everywhere continuous with x , for a certain enumerable and everywhere dense set of x -values (12). These correspond to the primary points) $f'(x) = 0$, and $f''(x)$ is defined (a value > 0) on a certain set and everywhere else on a set of x -values $f'(x) = f''(x) = 0$ for another x -set $\neq f'(x)$ not equivalent to $f'(x)$ or identical > 0 (see p. 37 to 38). (3) $f(x)$ is continuous everywhere through out where f' is everywhere zero and everywhere else $f'(x) = x$ and the logarithmic derivative $f''(x) = \frac{1}{x}$ the progressive $f''(x) = x$ for an everywhere dense and unenumerable set of x -values (13) $f'(x) = f''(x) > 0$ for a set of x which $\neq f'(x) = f''(x) = 0$ for a set of x (14) $f'(x) = f''(x)$ for a finite set of the same x (15) $f'(x)$ and $f''(x)$ do not exist for a set of x (16) $f'(x) = 0$ or $f''(x)$ does not exist (17) (18) and (19). The first set in the above includes the primary points.

An interesting question arises. As in the case of each function of Broder f is a primary point and therefore a point of non-differentiability for $u, f(u+0)$ is non-existent, what can be said about the existence of $\theta(+0)$ and $\theta'(+0)$?

The answer to this question may be attempted as follows:

(a) For each of the first two functions $f(x) = x^2$ which has been investigated $f'(x)$ in the preceding paragraph exists is finite and greater than 0 hence in the case of each f in Σ result $\theta(+0)$ exists and equals

1. In the case of the third function of Broder $f(x) = x$ and it is difficult to prove that $\theta(+0)$ exists.

(b) As regards $\theta'(+0)$ its existence is unlikely for every neighbourhood of 0 over as many functions where with f non-differentiable are everywhere done.



FIFTH LECTURE

THE FUNCTIONAL NATURE OF θ (CONTINUED).

§ 31

51. Just as the last lecture was devoted to the nature of the functional nature of θ as a function of h , so in this lecture we will discuss chiefly the functional nature of θ as a function of x . Before going on, however, to the results of Fréchet's researches relating to θ as a multiple valued function, I reproduce in abstract of their latest work interest the following passage from the first important contribution to the subject, viz. H. Fréchet's paper "On a fonction qui satisfait à la loi de la moyenne":—

(a) "The formula

$$f(x+h) - f(x) = f'(x)h + \theta(x, h) \quad \text{with} \quad \theta(x, h) < \epsilon$$

is known to every student of mathematics under the title of the mean value theorem. It is readily proved that the formula is valid if $f(x)$ is defined in an interval $a < x < b$ and if the derivative $f'(x)$ exists in the continuum $a < x < b$.

The quantity θ which occurs in the formula is evidently a function of two independent variables x and h and is defined for the values of h for which x and $x+h$ both lie in the interval in which $f(x)$ exists. The purpose of the present paper is to discuss the properties of this quantity θ , and we shall write

$$\theta = \theta(x, h)$$

whenever it is desired to emphasize the functional character.

In studying the function $\theta(x, h)$ it is at first immediately evident that $\theta(x, h)$ is continuous, the derivative θ_x is actually continuous, and. A slight inspection would tend to convince one that the converse is

* *Annales de Mathématiques*, Vol. 1, July 1894, p. 127, and Vol. 2, p. 127.

This paper has appeared only once before in the *Annales de Mathématiques*, by the writer at the commencement of the 20th century. It is now being reprinted in the *Annals of Mathematics*, Vol. 1, No. 1, 1911. The function θ is the θ of the *Annales de Mathématiques* by the writer at the beginning of the 20th century. It is now being reprinted in the *Annals of Mathematics*, Vol. 1, No. 1, 1911, to p. 127 of the *Annals*.



(ii) *Theorem:* The roots of the equation in θ ,

form either a finite or an infinite denumerable and nowhere dense set.

Proof.

Let f be a continuous function on $[a, b]$. Then according to a known theorem of Bolzano,

if f assumes both positive and negative values, every continuous function forms a finite or enumerable set of the values of t for which

$$F(t) = f(t) - \int_a^b f(t) dt = 0$$

form an everywhere dense set. In fact, if f is not identically zero, then there is at least one point where f is not zero.

For, when f is not identically zero, the set of points where f is not zero is everywhere dense.

For, if the set were nowhere dense, any point could be found open as a limit point of the set of points where f is not zero, and would therefore itself satisfy the condition. Therefore, any number in the interval $[a, b]$ will be the limit of a sequence of values of t where f is not zero.

Let t_1, t_2, \dots be a sequence of values of t satisfying (1) in $[a, b]$, arranged in order of their magnitude, and approximated by $u_1, u_2, \dots, u_n, \dots$, taking the greatest of the values. These values u_1, u_2, \dots are the points where f is not zero, and u_1 is the least value of t for which $f(t) = 0$, the single-valued function of t .

§ 33

§ 33. The following examples illustrate the theorem and definition of § 32.

Example 1. Let

$$f(x) = \frac{1}{2} (x^2 - 1)^2$$

Then $f(x) = 0$ for $x = -1, 1$, and $x = 0$. Therefore, the equation

$$f(x) = 0$$

has three roots, and the function $f(x)$ is not identically zero.

$$f(x) = \frac{1}{2} (x^2 - 1)^2$$

* See p. 14. The function $f(x) = \frac{1}{2} (x^2 - 1)^2$ is a continuous function of x on the interval $[a, b]$.

Therefore θ is given by the transcendental equation

$$\cos \left(\frac{1}{2} - \frac{1}{2} \log_{\theta} \frac{1}{h} \right) = \frac{1}{2} \cos \left(\frac{1}{2} \log_{\theta} \frac{1}{h} + \pi \right).$$

Taking θ to be the numerical least root whose cosine equals the right side of the above equation and setting $N(h)$ a suitable integer dependent on h , we have

$$\frac{1}{2} \log_{\theta} \frac{1}{h} = 2N(h)\pi \pm \alpha,$$

i.e., taking h to be positive,

$$\theta = \frac{1}{h} e^{-2N(h)\pi}.$$

where the integer N is chosen such that $0 \leq N < 1$.

If N_1 is any such integer for a given h , it follows that $N_1 + 1$, $N_1 + 2$ are all such integers. For instance if h is $e^{-2\pi}$ then N_1 is an integer.

$$\frac{1}{2} \cos \left(\frac{1}{2} \log_{\theta} \frac{1}{h} + \pi \right) = \frac{1}{2} \cos \pi$$

thus

$$\frac{1}{2}$$

and

$$\theta = \lim_{N \rightarrow \infty} \frac{1}{h} e^{-2N\pi} = 0.$$

The principal value θ_1 is e^{-2} , the other values a order of magnitude are

$$e^{-2+2\pi}, e^{-2-2\pi}, e^{-2+4\pi}, e^{-2-4\pi}, \dots$$

Example II. Take

$$f = \int_0^1 \cos \frac{1}{t} dt.$$

Then f exists for every value of t including 0.

Therefore the mean value theorem (1) of §2 applies

$$f(h) = u - f(\theta h) = h \cos \frac{1}{\theta h}.$$

Therefore θ is given by the transcendental equation

$$\cos \frac{1}{\theta h} = \frac{f(\theta)}{h},$$

in which the right-hand side is a constant $\frac{1}{h}$ ($f(\theta)$ varies with θ). Taking

h to be the number, say least integer, which exceeds the right side of the above equation, we have

$$\frac{1}{\theta h} = 2N(k) \pm \frac{1}{2},$$

$$\text{i.e., } \theta = \frac{1}{h(2N \pm \frac{1}{2})}.$$

where N is any integer chosen that θ lies between 0 and 1.

If N_1 is such an integer, then $N_1 - 1, N_1, N_1 + 1$ are all such integers.

For instance, if $h = \frac{1}{2m\pi + \frac{1}{2}}$, where m is any integer

$$\frac{1}{h} = \cos \theta = \frac{1}{2m\pi + \frac{1}{2}},$$

then θ nearly $\frac{1}{2m\pi + \frac{1}{2}}$

and $N = \left(\frac{1}{2m\pi + \frac{1}{2}} \right)$ nearly

$$\frac{1}{2m\pi + \frac{1}{2}}$$

The principal value θ_1 is nearly the other values

$$\frac{1}{2m\pi + \frac{1}{2}}, \frac{1}{2m\pi + \frac{3}{2}}, \frac{1}{2m\pi + \frac{5}{2}}, \dots$$



in order of magnitude, are nearly

$$\frac{2m\pi + \frac{\pi}{2}}{2(m+1)\pi - \left(\frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}}\right)} \quad \frac{2m\pi + \frac{\pi}{2}}{2(m+1)\pi + \frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}}}$$

$$\frac{2m\pi + \frac{\pi}{2}}{2} \quad \frac{2m\pi + \frac{\pi}{2}}{2}$$

$$\frac{2m\pi + \frac{\pi}{2}}{2(m+1)\pi - \left(\frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}}\right)} \quad \frac{2(m+1)\pi + \frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}}}{2}$$

§34

51. That the probability θ is not necessarily continuous at $h=0$ is proved by the proposition $\theta(h, 0) = 0$ in the case of Example I of § 33.

Proof. Let h tend to 0 by taking the values of the form $e^{-m\pi}$ where m is positive and integral, then as shown in connection with the study of Example I in § 33, $\theta_1(h)$ is always $e^{-m\pi}$. Thus

for this mode of approach $\theta(h, 0) = \theta_1(h)$ tends to $e^{-m\pi}$.

If h is of the form $\left(\frac{1}{2} + i\frac{\pi}{2}\right)^m$, where m is positive and integral, then

$$\frac{1}{2} \cos \left(\frac{1}{2} \log \frac{1}{f} + \frac{\pi}{4} \right) = \frac{1}{2} \left(\cos 2m\pi + \frac{\pi}{2} + \frac{\pi}{4} \right)$$

Therefore $\frac{1}{2}$ and $\theta(h)$ is equal to $\frac{1}{2} + \frac{1}{2}e^{-m\pi}$.

Thus for this mode of approach to 0 for h is by taking values of the form

$$\left(\frac{1}{2} + i\frac{\pi}{2}\right)^m$$

$\theta(h)$ tends to $\frac{1}{2}$.

Therefore $\theta_1(h)$ is not constant

§ 15.

3. That the problem of the value of f at any point $a+h$ is not necessarily differentiable there is proved as follows: Example 1. $f(x) = 1/x$.

In this case in the neighborhood of $a=1$ we observe that

$$-h^2 \sin \frac{1}{h}$$

Therefore, as shown in § 33,

$$\theta(h) = \frac{1}{(2N\pi \pm 1)}$$

and $\theta_1(h)$ tends to 1 as h tends to 0. Hence we see that $\theta_1(0)$ is not proved to be $\theta_1(0)$ as continuous at $h=0$.

We now proceed now to prove that $\theta_1(0)$ is not existent.

Proof:

(a) Consider

$$\theta_1(h) = 1$$

for the values of h of the sequence

$$\left\{ \frac{1}{2N\pi \pm 1} \right\}$$

Then we have in the equation

$$\theta = \frac{1}{(2N\pi \pm 1)} = 1$$

and the form

$$\frac{1}{2N\pi \pm 1} = \frac{1}{2} \quad \text{higher powers of } \frac{1}{2N\pi \pm 1} \text{ than the first}$$

Therefore $\theta(h) = \frac{2N\pi \pm 1}{2}$

and

$$\theta(h) = 1 = \left(\frac{2N\pi \pm 1}{2} \right) \left\{ \left(\frac{1}{2N\pi \pm 1} \right)^2 + \text{higher powers of } \frac{1}{2N\pi \pm 1} \text{ than the first} \right\}$$

1. Case is nearly equal to $-h \sin \frac{1}{h}$ as $h \rightarrow 0$

$$-h \sin \frac{1}{h} \approx -h \left(\frac{1}{h} - \frac{1}{6h^3} + \dots \right)$$

Hence for the sequence under consideration

$$\lim_{h \rightarrow +0} \frac{\theta_1(h) - 1}{h} = 0.$$

(b) Again consider

$$\theta_1(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$$

for the values of the sequence

$$\left\{ \frac{1}{(2m+1)\pi} \right\}$$

Then we have θ of the form

$$\sum_{n=0}^{\infty} \text{higher powers of } \frac{1}{(2m+1)\pi} \text{ than the last}$$

Therefore

$$\theta_1(t) = \frac{1}{(2m+1)\pi} + \dots$$

and

$$\frac{\theta_1(h) - 1}{h} = (2m+1)\pi \left\{ \frac{1}{(2m+1)\pi} + \text{higher power of } \frac{1}{(2m+1)\pi} \text{ than the first} \right\}.$$

Hence for the sequence under consideration

$$\lim_{h \rightarrow +0} \frac{\theta_1(h) - 1}{h} = \infty$$

which is different from the limit obtained in (a).

Therefore

$$\lim_{h \rightarrow +0} \frac{\theta_1(h) - 1}{h}$$

is not existent and consequently $\theta_1 u$ has no differential coefficient at $h=0$.

§ 30.

If we follow the statement made in passing in § 29 from Heine's paper that even if f is continuous at a it need not be continuous at a , the statement has been at least so stated as to leave h not distinguishing between the different values of ξ for a given h .

If it is to be any other way, then the corresponding case, namely, what if not f is continuous at a , Art. I of my first paper¹ is quite applicable to ξ .

Thus, f is continuous at a if and only if for any value of ϵ other than 0

there is a δ such that $|f(x) - f(a)| < \epsilon$ for any x such that $|x - a| < \delta$. To make no account of Heine's statement's error, let us examine his example

$$f(h) = h^2 \left(1 + \frac{1}{h}\right) = h + 1$$

$$f(0) = 0$$

Let us take $\epsilon = 1$ a point of discontinuity of the simple valued function f . The limit sequence of discontinuities of the first kind for f for h were for any sequence $\{h\}$ tending to 0 the corresponding sequence $\{f(h)\}$ does not tend to $f(0) = 0$ but to $f(h) = h + 1$ from $f(1)$. Thus

$$\lim_{h \rightarrow 0} f(h) = f(1), \quad \lim_{h \rightarrow 0} f(h) = f(1).$$

in which for any value h there are two different values of $f(h)$ of f . Therefore f is not a single valued function as is absurd.

Again f is not a point of discontinuity of the second kind for $h = 0$ for if it were there must be a sequence $\{h\}$ tending to 0 the sequence $\{f(h)\}$ corresponding to which does not tend to any limit. Therefore for a neighborhood of $h = 0$ as small as we please there must be values of ξ such that $\xi_1, \xi_2, \dots, \xi_n, \dots$ differing from one another by more than any arbitrarily chosen quantity > 0 . But $f(\xi_1), f(\xi_2), \dots, f(\xi_n), \dots$ are different from one another by a quantity as small as we please. Hence of the

¹ See the second footnote on p. 50.

continuity of $\frac{f(h)}{h}$ at h . Therefore

$$\begin{aligned} \phi(h) &= \left(1 - \frac{1}{n}\right) - \frac{1}{n} \phi(h) \\ &= \left(1 - \frac{1}{n}\right) - \frac{1}{n} \phi(h) \end{aligned}$$

different from one another by a quantity as small as we please. It is not possible if $\phi(h) = 1 - \frac{1}{n}$ for h in one set and by any other value in another.

§ 37

We proceed now to consider a new type of function, a study of the interesting case of a function which is nowhere differentiable. Take

$$f(x) = \sum_{n=1}^{\infty} a_n \phi\left(\frac{x}{b^n}\right)$$

where $\phi(x)$ is a continuous, but nowhere differentiable function, any Weierstrass's function

$$\phi(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{x}{b^n}\right)$$

Then the preceding value of $\phi(x)$ is a everywhere continuous but nowhere differentiable function.*

Proof

(a) That $\phi(x)$ is single valued is obvious but it is continuous for every value of x is to be seen in the following. Let $\phi(x)$ be equal to

$$\frac{f(h)}{h} = \phi\left(\frac{h}{b}\right).$$

For the sake of convenience we may assume $b = 1$. (b) that ϕ being a continuous function of its argument the argument $\frac{h}{b}$ and consequently $\phi\left(\frac{h}{b}\right)$ must be continuous.

(c) The non-differentiability of ϕ follows from the fact that ϕ is the equation

* See Prasad's cited paper.

$$f(\xi) = h$$

where ξ starts with a , h the right-hand side difference (for every value of h). Therefore

$$f'(h) = \omega(\xi) + h \frac{d}{dh} \{\omega(\xi)\}.$$

$$\text{Then } \frac{f'(h)}{h} = \frac{\omega(\xi)}{h} + \frac{d}{dh} \{\omega(\xi)\}.$$

$$h \rightarrow 0$$

But $\frac{f'(h)}{h}$ cannot exist if $\frac{d}{dh} \{\omega(\xi)\}$ would exist being equal

$$\frac{f'(h)}{h} = \frac{f'(h)}{h} \quad \text{The reference to } \frac{d}{dh} \{\omega(\xi)\} \text{ does not exist. It is proved that } \xi$$

and consequently $\frac{d}{dh} \{\omega(\xi)\}$ does not exist.

{ 34

(a) The next step in the study of the above differential θ_1 is to find the θ_1 value for each h corresponding to any given value of h . It has actually been found that θ_1 for the value $h = \frac{1}{2}$ is the formula $\theta_1 = \frac{1}{2}$. Whatever positive integer may be there is a value of θ between

$$\frac{1}{2} + \frac{1}{13^k} \text{ and } \frac{1}{2} + \frac{1}{13^{k+1}},$$

and another value between

$$\frac{1}{2} - \frac{1}{13^k} \text{ and } \frac{1}{2} - \frac{1}{13^{k+1}}.$$

* See the 3rd paper in which the value of θ_1 is carefully treated. The value of θ_1 is $\frac{1}{2}$ $\theta_1 = \left(\frac{1}{2} + \frac{1}{13^k} \right)$ for $k = 1$. Therefore θ_1 is given by $\frac{1}{2}$ or $\frac{1}{2} + \frac{1}{13^k}$ for $k = 1$.

Let θ_1 be a root of the function $\frac{1}{2} + \frac{1}{13^k}$ or $\frac{1}{2} - \frac{1}{13^k}$.

In the 3rd paper, the value of θ_1 is given as $\frac{1}{2}$ and the value of θ_1 is given as $\frac{1}{2} + \frac{1}{13^k}$ or $\frac{1}{2} - \frac{1}{13^k}$ in the 3rd of the paper, for "the zero" and "curve."

§ 40

1. We can illustrate the above theory by answering the following question: Is it true that, corresponding to every prescribed function $\phi(h)$ in case $n=1$ is a single-valued or the n -valued, prescribed as $\phi(h)$, n -valued $f(h)$ a multiple-valued function $f(h)$ for which the mean-value theorem

$$\phi(h) = h f'(h\theta)$$

$$\phi(h) = h f'(h\theta_0)$$

holds and f is a function? If so, what conditions must be imposed by the prescribed function ϕ or the prescribed set $\{h, \phi(h)\}$ in order that $f(h)$ may exist?

Let us consider the question of f as a single-valued function.

(a) If we assume that $\phi(h)$ is expandable in the form $\phi(h) = a_0 + a_1 h + a_2 h^2 + \dots$ and $\phi(h)$ is continuous, our search for $f(h)$ is only a h -series expansion in the form

$$f(h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \dots, \text{ to infinity,}$$

then take f together with ϕ as a function Λ and A_1 is to be zero we have by the mean-value theorem

$$f(h) = h f'(\theta h),$$

$$\sum_{k=1}^{\infty} A_k h^k = h \left\{ \sum_{k=1}^{\infty} k A_k (\theta h)^{k-1} \right\}.$$

If we put $\theta = 1$, the coefficients of h powers of h we get to value out the A_k as $\theta = 1$ is a special case of θ and $\theta = 1$ is not the case.

$$\left(\begin{matrix} 1 \\ 1 \end{matrix} \right) = 1 \text{ if } \theta = 1 \text{ and } A_1 \neq 0 \text{ if } \theta = 1 \text{ if } A_2 = 0 \text{ but } A_1 \neq 0$$

must be $\left(\frac{1}{\theta} \right)^2$, generally, if

$$A_2 = A_3 = \dots = A_{n-1} = 0 \text{ but } A_n \neq 0,$$

then $\theta = \left(\frac{1}{m} \right)^{1/n}$. With this important condition the A_k are not eliminated and expressible in terms of ϕ as $\theta = 1$ if $A_1 \neq 0$.

$$A_1 = \theta_1 A_2, A_2 = \theta_2 A_3, \dots, A_{n-1} = \theta_{n-1} A_n = 2A_n(\theta_1 + \theta_2 + \dots + \theta_{n-1}) + 2A_n(\theta_1 \theta_2 + \theta_1 \theta_3 + \dots + \theta_1 \theta_{n-1}) \times \dots$$

(b) The general restriction on the properties of the single-valued $\phi(h)$ are of this form ($h > 0$) that ϕ is continuous and ϕ that it is not a ϕ -series where n is different from

It is thus clear from (1) and (2) that θ may exist but θ' may not exist corresponding to $\theta(h)$, θ and θ' not being unique θ and θ' for a value of h . θ must fulfil certain restrictions and cannot be arbitrary. The following examples will illustrate this remark.

Ex. 1. If θ is constant θ can have values only of the form $\left(\frac{1}{n}\right)^{-1}$ where n is a positive integer.

Ex. 2.4. There is no $\theta(h)$ corresponding to $\theta' = \frac{1}{h^2}$ for $h = \frac{1}{2}$.

Ex. 3. $\theta(h) = A_1 \left\{ 1 + \frac{1}{h^2} + \frac{1}{h^4} + \frac{1}{h^6} + \dots \right\} + A_2$ where A_1 is any constant different from 0.

$$\text{For } \theta = \frac{1}{\sqrt{3}} + \frac{1}{3}, f(h) = A_1 \left\{ h^2 + \frac{6}{3\sqrt{3}-4} h^4 + \dots \right\}$$

A_1 being any constant different from 0.

Ex. 4. If $\theta = \frac{1}{h^2} + \frac{1}{h^4} + \frac{1}{h^6} + \dots$ then for each value of h value $f(h) = h^2 + h^4 \sin \frac{1}{h^2}$.

Ex. 5. If $\theta^2 = \frac{2}{3} - 3h^2 \left(\frac{2}{3} \right) + \frac{1}{2} \frac{1}{h^2}$ for small values of h , then for each value of $h = \frac{2}{1} + h^2 + \dots$ and $\frac{1}{h^2}$.

Ex. 6. If

$$\theta = \left[1 + \frac{1}{h^2} \cos \left\{ \log \frac{1}{h^2} + \frac{1}{h^2} \right\} \right] / \left[2 + \left\{ \left(\log \frac{1}{h^2} + \frac{1}{h^2} \right) \right\} \right]$$

then $f(h)$ will be of the form $C^2 + C D^2 \cos \left\{ \log \frac{1}{h^2} + D \right\}$ C and D being arbitrary constants and in fact $C = \frac{1}{5}$, $D = \tan^{-1} \frac{1}{2}$.

61. Let us consider now the case in which $\theta(h)$ is multiple valued

Example 1. Let $f(x) = x^2$ and $a = 0$. Then $f'(x) = 2x$. The function $f(x)$ satisfies the conditions of the theorem. Find ξ for $h = 1$.

Solution. If $f(x) = x^2$, then $f'(x) = 2x$ when $h = 1$ and $a = 0$. The value of ξ in $(0, 1)$ equal to

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1^2 - 0^2}{1 - 0} = 1$$

is $\xi = 1$. This is the only value of ξ in $(0, 1)$ which satisfies the condition that it is a power series in h .

Example 2. Let $f(x) = x^3$ and $a = 0$. Then $f'(x) = 3x^2$. The function $f(x)$ satisfies the conditions of the theorem. Find ξ for $h = 1$. The value of ξ in $(0, 1)$ equal to

Solution. If $f(x) = x^3$, then $f'(x) = 3x^2$ when $h = 1$ and $a = 0$. The value of ξ in $(0, 1)$ equal to

Example 3. Let $f(x) = x^4$ and $a = 0$. Then $f'(x) = 4x^3$. The function $f(x)$ satisfies the conditions of the theorem. Find ξ for $h = 1$.

Solution. If $f(x) = x^4$, then $f'(x) = 4x^3$ when $h = 1$ and $a = 0$. The value of ξ in $(0, 1)$ equal to

Example 4. Let $f(x) = x^5$ and $a = 0$. Then $f'(x) = 5x^4$. The function $f(x)$ satisfies the conditions of the theorem. Find ξ for $h = 1$.

Solution. If $f(x) = x^5$, then $f'(x) = 5x^4$ when $h = 1$ and $a = 0$. The value of ξ in $(0, 1)$ equal to

Hence $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$.

Then values of the ξ 's satisfy the second terminal value $f(1)$ defined as $f(1) = 1$ if $\xi = 1$ or $f(1) = 0$ if $\xi = 0$ or $\frac{1}{2}$ or $\frac{3}{4}$ or $\frac{1}{4}$ or $\frac{3}{2}$. And

$$f(h) = \frac{h^2}{2} \text{ or } h - \frac{h^2}{2} - \frac{1}{2} \text{ according as } h \text{ is in } (0, \frac{1}{2}) \text{ or } (\frac{1}{2}, 1).$$

Ex. 2. If ξ is given by the following scheme, find $f(h)$ on the same assumption as in Ex. 1.

Dividing the interval $(-1, 1)$ into four quarters $\left(-1, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\left(\frac{1}{2}, 1\right)$, $\left(1, \frac{3}{2}\right)$ and calling them respectively the first, second, third

and fourth quarters, ξ is equal to $\frac{h}{2}$ when h is in the first quarter, for values of h in the second, ξ is $-\frac{1}{2}$ or $\frac{1}{2}$ or $\frac{3}{4}$ or $\frac{1}{4}$ or $\frac{3}{2}$.

$$\frac{-\frac{h^2}{2} + \frac{1}{2}h - \frac{1}{2}}{h}.$$

According as ξ is in the first or third quarter $f(\xi) = 1$ or $f(h)$ in the third quarter ξ , $\frac{1}{4} - \xi$ or $\xi - \frac{3}{4}$ equals

$$\frac{\frac{h^2}{2} - \frac{1}{2}h + \frac{3}{2}}{h}$$

according as ξ is in the first or third or third quarter, and lastly for values of h in the fourth quarter $\xi = \frac{1}{2}$ or $\frac{3}{4}$ or $\frac{1}{4}$ or $\frac{3}{2}$.

$$\frac{\frac{h^2}{2} - \frac{1}{2}h + \frac{3}{2}}{h}$$

according as ξ is in the first, second, third or fourth quarter.

(b) For the case in which $\theta(h)$ has a finite number of values see Prasad's paper 'On the determination of $f(h)$ corresponding to a given Riesz function $\theta(h)$ when θ is multiple-valued' Proc. Ramanujan M. S., Vol. XII).

SIXTH LECTURE

ROLLE'S THEOREM AS A FUNCTION OF x AND h , THE MEAN VALUE
THEOREM, GENERALIZED FUNCTIONS θ

§ 41

Today I will first fix as the independent function θ as a function of x and h and then proceed to give a number of theorems and results each of which is in some sense connected with the mean value theorem for the function θ as a function of x and h . I proceed to formulate and prove in the words of Little the following three theorems.

Let $f(x)$ be a function of x such that $a \leq x \leq b$ and also $a \leq x+h \leq b$ further let $f(x)$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Then if $f(x)$ is a function of x satisfying the above conditions, $f(x+h) - f(x) = h f'(x + \theta h)$, where θ is a function of x and h such that $0 < \theta < 1$.

with a value θ independent of x and h such that $0 < \theta < 1$.

Proof

If x lies on the left side of a then is the left side and therefore not on the right side of M_1 differentiable with respect to x consequently is not differentiable because of the condition $f(x)$ is twice differentiable. By differentiating both sides of M_1 with respect to x and with respect to h we have

$$(1) \quad \begin{cases} f(x+h) - f(x) = h f'(x + \theta h), \\ f'(x+h) = f'(x) + \theta f''(x + \theta h). \end{cases}$$

From these two equations follow immediately

$$f(x+h) - f(x) = h f'(x + \theta h).$$

If differentiating the above equation with respect to h we have

$$(2) \quad f'(x+h) = f'(x) + \theta f''(x + \theta h).$$

Now $f'(x)$ is differentiable in the inside of a and b and therefore continuous and consequently according to (1) $f'(x)$ is continuous there. Consequently from (2) we have for $h \rightarrow 0$

$$f'(x) = f'(x + \theta h) = f'(x).$$

If now $f'(x)$ were identically equal to 0, then $f(x)$ would be a linear and the formula (M_1) would be satisfied for every arbitrary value of h . This trivial case shall be here, as also in the following pages, excluded.

Therefore θ must be $\frac{1}{2}$.

Thus from (2) we have

$$f'(x+h) = f' \left(x + \frac{h}{2} \right).$$

Since h is permitted to take $\frac{1}{2}$ or any other value for which $x + \frac{h}{2}$ and $x+h$ are in the range of f , it follows that $f'(x)$ must be a constant different from 0 and therefore $f(x) = ax^2 + 2bx + c$, a, b, c being constants.

On the other hand, if in (M_1) θ means a variable $\theta(h)$ independent of x and depending on h only, and further $\theta(h) \neq 0$ for every value of h not equal to 0, then of all the f -terms $f(x)$ which are continuous in (x) , the only one included and differentiable in (x) is the one which excluded $f(x) = c$ for $x \neq x_0$ alone has the property to satisfy (M_1) with such $\theta(h)$ as stated above, which is a linear function. Thus $\theta(h)$ is $\frac{1}{2h} \log \frac{f(x+h) - f(x)}{f(x) - f(x-h)}$.

Proof.

First, if f is linear, (M_1) by the same reasoning as before, has been satisfied for all h and $f(x) = c$ and therefore also $f'(x) = c$ for $x \neq x_0$. If f is not linear, with respect to x and with respect to h we have from (M_1) , putting ξ for θh

$$(3) \quad \begin{cases} f(x+h) = f(x) + h f'(x) \\ f(x+h) = f(x+\xi) + h f'(x+\xi) \xi, \end{cases}$$

where $\xi = \frac{df}{dh}$. One can assume that ξ is not identically zero, for otherwise

we would ξ be constant and from the above equation would also f be a constant which would lead to the trivial case of $f(x)$ as a linear function. Therefore, it follows from the two equations by the elimination of $f(x+h)$ as is easily seen, that

$$(4) \quad f(x+h)(1-\xi') + \xi' f(x) = f(x+\xi)$$

$$\text{or} \quad \xi' \{ f(x+h) - f(x) \} = f(x+\xi) - f(x)$$

Now however this equation can be also satisfied with respect to

x if $f'(x+h) = f'(x)$ for every values of h and x , then must $f(x)$ be constant in the whole range, a possibility which was excluded at the outset by excluding linear f . However, if for a definite value of h and x for every values of x , $f(x+h) - f(x) = \xi' f(x) - f(x)$



Consequently (1) admits of a solution with $0 < \xi < h$.

$$\xi = \frac{f(x+h) - f(x)}{f'(x+h) - f'(x)}$$

The right side which as a rule is not a dependent of x , is differentiated with respect to x . Therefore

Moreover f' as function of x is differentiated with respect to x

$$f'(x+h)(1-\xi) - f'(x+\xi)(h\xi' + \xi) = 0$$

or, by using (J),

$$(5_1) \quad f''(x+h)(1-\xi) - f''(x+\xi)(h\xi' + \xi) = 0,$$

and differentiating further the second member with respect to x

$$(5_2) \quad f'''(x+h)(1-\xi) - f'''(x+\xi)(h\xi'' + \xi') = 0.$$

Now however, as immediately clear, the determinant of the two equations (5₁) and (5₂)

$$\begin{vmatrix} f''(x+h) & f''(x+\xi) \\ f'''(x+h) & f'''(x+\xi) \end{vmatrix} = 0$$

if ξ cannot be zero, then necessarily $f''(x+h) = f''(x+\xi)$ and $f'''(x+h) = f'''(x+\xi)$ which clearly means that the previous considerations does not happen. If we take $\xi = 0$, we obtain as a consequence of the equations (5) we have recently the last differential equation admits of being written in the form

$$\frac{f''(x+h)}{f'''(x+h)} = \frac{f''(x+\xi)}{f'''(x+\xi)},$$

since $\xi = \theta h$ and $0 < \theta < 1$ we make of $\theta < 1$ therefore we may put

$$\xi = h + p \cdot h$$

where p is $\neq 1$. Therefore the problem is reduced to with $x+h$ changing in

$$(6) \quad \frac{f''(z)}{f'''(z)} = \frac{f''(z+p)}{f'''(z+p)}.$$

If p must be an absolute constant for otherwise $p = 1$ ($\xi = h$) and therefore by (1) $f(x+h) = f(x) + hf'(x)$ which would again lead to the case

had the period h chosen more convenient, that is $f(x+h) = f(x) + \theta h f'(x)$ would also have $f(x+h) = f(x) + hf'(x)$. By repeatedly proceeding as above it will follow that f' has an everywhere constant and must therefore be constant. Since follows that $f(x+h) = f(x) + hf'(x)$ with $f' = \text{constant}$.



of f near $f(x)$ —therefore $f = f(x) + \epsilon$ and $x = x + \delta$ by differentiation and therefore contains x in some sense for $f(h)$ and therefore you can take all the values of x certain and then. This leads however with respect to the equation $f(x) = f(x)$ the result that $\frac{f'(x)}{f'(x)}$ must be natural. If n writes x in the place of x

$$\frac{f''(x)}{f''(x)} = C$$

where C denotes a constant. If $C \neq 0$, this equation has a unique solution $f(x) = Cx^2$ for $C \neq 0$.

$$f(x) = Cx^2 \quad \text{for } C \neq 0.$$

$$f(x) = C_1 x^2 + C_2 x + C_3 \quad \text{for } C = 0;$$

C_1, C_2, C_3 being constants.

Leaving aside the case of $C = 0$ which gives θ independent of x , we have the expression $f(x) = Cx^2 + C_2 x + C_3$ with $C \neq 0$.

17. Theorem III. (1) If f the function $f(x)$ which are continuous in (1), (2) the only being needed and sufficient condition for f being excluded there is no function for which the theorem M_1 is satisfied with a

θ independent of x expressed in $f(x) = Cx^2 + C_2 x + C_3$ for $C \neq 0$.

Proof: "The left side of the equation

$$\frac{f(x+h) - f(x)}{h} = f'(\xi), \quad (\xi = x + \theta h)$$

is for $h \neq 0$ and for every fixed value of x in the interval $a < x < b$ continuous and differentiable with respect to h and since the argument ξ as a function of h is continuous and differentiable with respect to h for every fixed value of x , therefore the same holds for $f'(\xi) = f'(\xi)$ exists in the interval $a < x < b$. Moreover, $f(x) = Cx^2 + C_2 x + C_3$ with respect to h and with respect to x given the law $f(x) = Cx^2$.

$$f(x+h) = f(x + \theta h) + h f'(x + \theta h) \quad "$$

$$f(x+h) - f(x) = h f'(x + \theta h)(1 + \theta h)$$

From the first it follows at once because of $0 < \theta < 1$ that for $h \neq 0$ also $f'(x + \theta h)$ is continuous as $f'(x)$ is continuous for $a < x < b$.



In the second equation the factor of $(1-\theta)h$ cannot vanish for every value of h and consequently $1-\theta$ admits of being eliminated and one obtains thus

$$(1+\theta h-\theta)f(x+h)+\theta f'(x)=(1+\theta h)f(x+\theta h).$$

One more differentiation with respect to h gives

$$\theta f'(x+h)+(1+\theta h)-\theta f'(x+\theta h)=(1+\theta h)f'(x+\theta h)-\theta$$

Now θ is independent of h and converges to 1. Thereby one obtains by having regard to the continuity of $f'(x)$

$$(1-\theta)f'(x)=f''(x)(1-\theta)$$

and since the factor $(1-\theta)$ vanishing of $f'(x)$ would lead to linear $f(x)$ there remains the only possibility

$$\theta=1$$

i.e., θ is the same constant as in Theorem 1."

§ 42

Now I propose to now to give briefly the treatment of the question: What conditions must be satisfied by θ as a function of x and h in order that there should be a mean value function $f(x)$ to satisfy the mean value theorem

$$f(x+h)=f(x)+h f'(x+\theta h)?$$

Let θ be a function subject to the condition $0<\theta<1$ of the two independent variables x and h and x and $x+h$ which take values inside the quadratic region

$$a\leq x_1\leq b, a\leq x_2\leq b,$$

where f not only $f(x_1, x_2)=f(x+\theta(x_1, x_2)(x_2-x_1))$ takes only values of the interval $a\leq f\leq b$. The question is: What conditions must be satisfied by θ as a function of x_1, x_2 in order that there be a function $f(x)$ which is continuous for $a\leq x\leq b$, differentiable for $a<x<b$ and satisfies the equation

$$(M) \quad f(x_2)-f(x_1)=(x_2-x_1)f'(x+\theta(x_1, x_2)(x_2-x_1)).$$

Let those pairs x_1, x_2 be excluded for which the following conditions I-IV are not satisfied and let the remaining part of the region be denoted by B

$$I. \quad x_1 \neq x_2, \quad i.e., \quad h \neq x_2 - x_1 \neq 0$$

II. $\theta(x_1, x_2)$ possesses the continuous partial differential co-

$$\text{coefficients } \frac{\partial \theta}{\partial x_1} = \theta_1, \quad \frac{\partial \theta}{\partial x_2} = \theta_2, \quad \frac{\partial \theta_1}{\partial x_2} = \frac{\partial \theta_2}{\partial x_1} \cdot \theta = \frac{\partial \theta_{12}}{\partial x_2} \\ = \theta_{122}$$

It is easily seen that the function Π is also annihilated by ξ_1 and ξ_2 . The partial differential coefficients $\frac{\partial \xi}{\partial x_1} = \xi_1$, $\frac{\partial \xi}{\partial x_2} = \xi_2$ cannot be identically 0, for otherwise ξ would be a constant and therefore $\phi = 0$ which would give according to (M) a linear $f(x)$ which case was excluded beforehand.

Now the left side of (M) and consequently also the right side are partially differentiable with respect to x_1 as well as x_2 and because by II, ξ_1 and ξ_2 are existent therefore $f'(\xi)$ must exist and equal $\frac{\partial f(\xi)}{\partial x_1} / \xi_1 = \frac{\partial f(\xi)}{\partial x_2} / \xi_2$ it being assumed that ξ_1 and ξ_2 are both different from zero. Under this supposition by the partial differentiation of (M) we have

$$(M) \quad \begin{cases} f'(\xi) = (x_2 - x_1) f'(\xi) \xi_2 + f'(\xi) \\ f'(\xi) = (x_2 - x_1) f'(\xi) \xi_1 + f'(\xi) \end{cases}$$

and hence because of I

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1) \xi_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

III. Neither ξ_2 nor ξ_1 vanish.

The expression for $f'(\xi)$ admits of partial differential coefficients in the whole of the region (B) and because of

$$f'(\xi) = \frac{\partial f(x_1)}{\partial x_1} / \xi_1 = \frac{\partial f(x_1)}{\partial x_2} / \xi_2$$

$f''(\xi)$ is existent in the whole of (B). Now this is accomplished by differentiating with respect to x_1 both the sides of the first equation (M) or with respect to x_2 both the sides of the second equation we have

$$(M') \quad (x_2 - x_1) f'(\xi) \xi_{12} + (x_2 - x_1) f''(\xi) \xi_1 \xi_2 + f'(\xi) \xi_2 = \xi_2 = 0$$

where

$$\xi_{12} = \frac{\partial \xi_1}{\partial x_2} = \frac{\partial \xi_2}{\partial x_1}$$

§ 48.

79. The following theorems, the proof of each of which is briefly indicated, may be considered to be generalizations of the mean value theorem in the sense that each of them depends for its validity on Rolle's theorem and gives the mean value theorem as a particular case.

(a) *Lagrange's remainder theorem*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h) \quad 0 < \theta < 1, \quad n \geq 1$$

Let $\psi(t)$ denote $f(b) - f(t) - (b-t)f'(t) - \frac{(b-t)^2}{2!} f''(t) - \dots - \frac{(b-t)^{n-1}}{(n-1)!} f^{(n-1)}(t) - \frac{(b-t)^n}{n!} P$,

$$\frac{(b-t)^{n-1}}{(n-1)!} f^{(n-1)}(t) - \frac{(b-t)^n}{n!} P,$$

where $b = x+h$ and P is independent of t and is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} P$$

Then $\psi(x) = 0$, $\psi(x+h) = 0$ and if $f^{(n)}(t)$ exists for every value of t made in $(x, x+h)$, Rolle's theorem gives

$$\psi'(t) = 0$$

for a value of t between x and $(x+h)$. But

$$\psi'(t) = \frac{(b-t)^{n-1}}{(n-1)!} \{P - f^{(n)}(t)\}.$$

Therefore we have

$$\psi(x+\theta h) = 0, \text{ i.e., } P = f^{(n)}(x+\theta h).$$

(b) *Cauchy's generalized mean value theorem*

$$\frac{\phi(x+h) - \phi(x)}{f(x+h) - f(x)} = \frac{\phi'(x+\theta h)}{f'(x+\theta h)} \quad 0 < \theta < 1.$$

Let

$$\psi(t) = \phi(t) - \phi(x) - \frac{\phi(x+h) - \phi(x)}{f(x+h) - f(x)} \{F(t) - F(x)\}$$

then $\psi(x) = 0$, $\psi(x+h) = 0$ therefore Rolle's theorem gives

$$\psi'(t) = 0$$

for a value of t between x and $x+h$ and $\phi'(t)$ and $f'(t)$ exist for every value of t made in $(x, x+h)$, and $f'(t)$ is nowhere 0 or infinite. Hence the theorem

(c) *Green's theorem* and *P. Lacroix's generalized mean value theorem*

$$\begin{vmatrix} f(x+h) & \phi(x+h) & F(x+h) \\ f(x) & \phi(x) & F(x) \\ f'(x+\theta h) & \phi'(x+\theta h) & F'(x+\theta h) \end{vmatrix} = 0, \quad 0 < \theta < 1.$$

Let

$$\psi(t) = \begin{vmatrix} f(x+h) & \phi(x+h) & F(x+h) \\ f(x) & \phi(x) & F(x) \\ f(t) & \phi(t) & F(t) \end{vmatrix},$$

then $\psi(0) = \psi(1) = 0$; therefore by the theorem given

$$\psi'(c) = 0$$

for some c between x and $x+h$ and $\psi'(c) = 0$ exists for every value of h in $(x, x+h)$ and $\psi'(c) = 0$ is a continuous function. Hence the theorem may be extended to the case where f and ϕ are generalized functions and F is a linear function of f and ϕ .

Let $f(x) = f_0(x) + f_1(x) + \dots + f_n(x)$ and $\phi(x) = \phi_0(x) + \phi_1(x) + \dots + \phi_n(x)$ we have

$$\begin{vmatrix} f_0(x_1) & f_1(x_1) & \dots & f_n(x_1) \\ f_0(x_2) & f_1(x_2) & \dots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & \dots & f_n(x_n) \end{vmatrix} = 0,$$

$$f_0(x_1) = f_1(x_1) = \dots = f_n(x_1)$$

where x is a mean of x_1, x_2, \dots, x_n .

(d) *Pompeiu's theorem* 2

$$\{f(x+h) - f(x)\} / \{1 - (x+h)^2\} = \{f(x) - f(x-h)\} / \{1 - (x-h)^2\}, \quad 0 < h < 1$$

The theorem is proved by P. U. on the basis of the fact that f and f' are positive and increasing.

$$f(x+h) - f(x) = \int_x^{x+h} f'(t) dt = \int_x^{x+h} f'(t) dt = \int_x^{x+h} f'(t) dt = \int_x^{x+h} f'(t) dt, \quad 0 < h < 1;$$

$\phi(u, v)$ standing for

$$\frac{\partial}{\partial u} \phi(u, v) = \frac{\partial}{\partial v} \phi(u, v) \quad \text{for } \frac{\partial}{\partial u} \phi(u, v) = \frac{\partial}{\partial v} \phi(u, v).$$

Under the function $\chi(t) = f(t) - \phi(t)$

$$\chi(1) = f(1) - \phi(1) = 0, \quad \chi(0) = f(0) - \phi(0) = 0.$$

Then by the mean value theorem the ratio

$$\chi(1) = \chi(0) + \chi'(c) = 0, \quad 0 < c < 1$$

at 1

$$\chi'(c) = \frac{\partial}{\partial t} \chi(c) = \frac{\partial}{\partial t} f(c) - \frac{\partial}{\partial t} \phi(c) = f'(c) - \phi'(c).$$

Since by assumption $f'(c) = \phi'(c)$ we have

$\chi'(c) = 0$ for every c in the interval $(0, 1)$. Hence $\chi(t) = 0$ for every t in the interval $(0, 1)$.

(a) As stated in the first lecture, Bolle and Hayashi have studied this θ and Bolle has given a number of results similar to those for Rolle's function θ .

Mr. Bolle has shown that if $\phi(x) = x^{n+1} - 1$ exists, is finite and $\neq 0$, $\theta(+0)$ exists and equals $\frac{1}{n+1}$.

(b) The number θ occurring in Cauchy's generalized theorem was apparently studied at Cambridge as early as 1864 for which year there is a question paper* containing the question

If $f(x)$ denote $\frac{f(x) - \phi(x)}{f'(x) - \phi'(x)}$ prove that

$$\theta = \frac{1}{2} + \frac{h}{24} f''(x) \text{ approximately when } h \text{ is small.}^{\dagger}$$

In the above θ is given by

$$\frac{f(x+h) - f(x)}{\phi(x+h) - \phi(x)} = \frac{f'(x) + \theta h}{\phi'(x) + \theta h}$$

(c) As Bolle considered the case in which Rolle's function $\theta(x, h)$ is an absolute constant, so Takahashi has considered the case in which the θ in Cauchy's generalized theorem is an absolute constant.[‡] He has proved that θ must be $\frac{1}{2}$ and that f and ϕ being assumed to be differentiable five times, they must belong to only the following three types

- (1) $f = Ax^3 + Bx + c, \phi = A'x^3 + B'x + c'$
- (2) $f = Ae^{rx^2} + Be^{-rx^2} + c, \phi = Ae^{rx^2} + B'e^{-rx^2} + c'$
- (3) $f = A \sin(px + q) + B, \phi = A' \sin(px + r) + B'$
or
 $f = A \cos(px + q) + B, \phi = A' \cos(px + r) + B'$

In the above $c = 0$ in (1), $c > 0$ in (2) and $c < 0$ in (3) — all the other constants are arbitrary but different from 0.

(d) Takahashi has considered a question similar to the above for the θ in Genocchi and Peano's theorem.[§]

* Cambridge International Mathematical Olympiad, 1964, p. 19, Q. 6, and 20.
[†] Cf. Trinity College, Hartford, Conn., 1864.
[‡] *Le Mat.*, pp. 127-130.
[§] *Le Mat.*, pp. 137-141.

APPENDIX A.

ON POMPEII'S PROOF OF THE MEAN VALUE THEOREM

UNIVERSITÉ DE BUCAREST,

le 15 Mars, 1931.

Très honoré Monsieur et Professeur,

Je viens de recevoir votre lettre ainsi que l'extrait de votre lettre du 15/15 vous me faites l'honneur de donner une page à ma méthode pour démontrer le théorème des accroissements finis.

Veuillez en accepter tous mes remerciements très sincères.

Vous faites aussi l'exposé de ma démonstration de quelques observations (criticism of Pompeii's proof) auxquelles vous désirez avoir aussi mon opinion.

Je suis très sensible à la détermination de ce procédé.

Voici quelle est mon opinion.

Le fait que vous signalez par votre exemple (page 28) est exact et de plus c'est un fait général dans le cas d'une fonction décroissante qui est à dire pourvu de points de discontinuité.

Mais ce fait n'intervient pas dans ma démonstration.

Dans ma démonstration le point c est toujours situé aux intervalles (x_1, y_1) et si ce fait signalé par votre exemple ne peut pas se produire.

Dans votre exemple le point $c = \frac{1}{2}$ est extérieur aux intervalles

$$x_1 = \frac{1}{2} + \frac{1}{2^{1/2}}, \quad y_1 = \frac{1}{2} + \frac{1}{2^{1/2}} - \frac{1}{2}$$

c'est pour cela que le fait signalé par vous est possible.

En résumé vous signalez par votre exemple un fait exact mais ce fait n'a rien à voir dans ma démonstration parce que les circonstances sont autres dans ma méthode le point c est toujours situé aux intervalles (x_1, y_1) et donc le fait en question ne peut pas se produire.

Et, puisque ainsi débattre la question présente un intérêt scientifique, j'ai rédigé une Note (ci-jointe) et si vous trouvez cela convenable vous pouvez (après traduction en anglais, ou même sans sa forme primitive) la faire figurer à la fin de votre œuvre comme une simple Note explicative.

J'y tiens en savoir très obligé pour intérêt scientifique de la question.
Veuillez agréer très honore Monsieur le Professeur l'assurance de
ma considération très distinguée et de respect.

voire tout dévoué

D. POURCEL

Ajuda à l'Université de Paris

NOTA

Sur une propriété des fonctions dérivées

par D. POURCEL.

1. Soit $f(x)$ une fonction définie dans un intervalle a, b et admettant pour tout point x , intérieur à (a, b) une dérivée bien déterminée $f'(x)$.

On a vu (il lire que) si x est un point fixe pris dans (a, b) le rapport

$$\frac{f(x+h) - f(x)}{h}$$

a une limite bien déterminée lorsque h tend vers zéro (pour valeurs positives ou négatives).

2. Nous prouvons ici qu'il y a un point pris dans l'intérieur de a, b et (x_1, y_1) une suite d'intervaux (x_1, y_1) de plus en plus petite et tendant vers le point c .

Formons les rapports

$$\frac{f(x_1) - f(y_1)}{x_1 - y_1} = R(x_1, y_1).$$

Que peut-on dire de la limite des rapports $R(x_1, y_1)$ lorsque ces intervalles (x_1, y_1) tendent vers le point c ?

Voici la réponse précise:

1° Cette limite peut ne pas exister du tout. On peut facilement trouver un exemple avec la fonction dérivée

$$f(x) = x \ln \frac{1}{x}$$

dans le voisinage du point $x = 0$.

2° Cette limite peut exister et être différente de la valeur de $f'(x)$ au point $x =$

(voir l'exemple signalé à la page 24 de ce livre)

3° Cette limite existe et est égale à la dérivée $f'(c)$ en tous les intervalles (x_1, y_1) contenant le point c à leur intérieur.

Let us recall immediately the identity

$$\frac{f(x_2) - f(y_2)}{x_2 - y_2} = \frac{f(x_2) - f(c)}{x_2 - c} + \frac{f(c) - f(y_2)}{c - y_2} \cdot \frac{c - y_2}{x_2 - y_2}.$$

Since c is interior to (x_1, y_1) we have

$$\frac{x_2 - c}{x_2 - y_2} < 1 \quad \left| \frac{c - y_2}{x_2 - y_2} \right| < 1$$

and thus the rapport du premier membre est ~~compris~~ *compris* entre les deux rapports qui figurent au second membre.

Il est ce fait seul qui se présente dans la démonstration du théorème que nous venons d'énoncer et que nous exposons aux pages 26-27 de ce livre.

3. Mais il reste à dire ce qui précède que la dérivée d'une fonction $f(x)$ peut être définie

soit par la limite du rapport classique

$$(1) \quad \frac{f(x+h) - f(x)}{h},$$

soit par le rapport

$$(2) \quad \frac{f(x') - f(x'')}{x' - x''}$$

avec la condition que le point x' est à un ϵ arbitrairement petit de la dérivée et que le point x'' est à un δ arbitrairement petit de x' .

$$x' - \epsilon < x < x' + \epsilon$$

Ainsi la dérivée cherchée est aussi bien la limite du rapport (1) que la limite du rapport (2).

D. POUSSIN

Remarks on the above

By

GANKER PRASAD

(I)

The fact that $\frac{f(x_2) - f(y_2)}{x_2 - y_2}$ tends to a limit as x_2 and y_2 both tend to c

even if for every value of x

$$x_1 < c < y_1$$

does not carry us far, as will be clear from the following examples, in each of which $h > 0$, $h \rightarrow 0$.

Example 1. Let $f(x) = \frac{1}{x}$ and take $x = y_1, x_1$ and $g = c$ in any manner whatsoever.

Then $f(x_1) - f(y_1)$ is zero for every value of h ; consequently x_1 and y_1 both tend to 0 while $\frac{f(x_1) - f(y_1)}{x_1 - y_1}$ also tends to 0.

But there is no differential coefficient for $f(x)$ at $x=0$, and as a matter of fact the mean-value theorem does not hold.

Example 2. Let $f(x) = \sin \frac{1}{x}$, so that

$$f(x) = \int_0^x \sin \frac{1}{t} dt;$$

and take $x_1 = y_1 = x$, tending to 0 in any manner whatsoever.

Contrary to Example 1. For given h a statement under which $\Delta f = f(0)$ exists ¹ and is 0.

Also $f(x_1) - f(y_1)$ as f is an even function, consequently, x_1 and y_1 both tend to 0 while $\frac{f(x_1) - f(y_1)}{x_1 - y_1}$ also tends to 0.

Also there is a differential coefficient for $f(x)$ at $x=0$, and as a matter of fact the mean-value theorem holds.

Example 3. Let $f(x) = x \sin \frac{1}{x}$ and take $x_1 = y_1 = x$, tending to 0 in any manner whatsoever.

Then $f(x)$ being an even function $f(x_1) = f(y_1)$ is zero for every value of h , consequently x_1 and y_1 both tend to 0 while

$$\frac{f(x_1) - f(y_1)}{x_1 - y_1}$$

also tends to 0.

Although there is no differential coefficient at $x=0$ the mean-value theorem holds ².

The series follows as follows:

$$f(x) = \int_0^x \left(1 - \frac{1}{n} + \frac{1}{2!} \right) dt = x^2 \cos \frac{1}{x} = \int_0^x x^2 \cos \frac{1}{t} dt.$$

Therefore $\cos \frac{1}{x} = 1 - \frac{1}{n}$ the sum of the series at coefficient 0 at $x=0$.

$$\cos \frac{1}{x} = \int_0^x \left(1 - \frac{1}{n} + \frac{1}{2!} \right) dt$$

even though the coefficient at coefficient 0 at $x=0$.

¹ See A. 1. 1.



(11)

Leaving now the Note of Professor Pompeiu given in this Appendix and going back to the proof of Theorem 15, I wish to emphasize that like the classical proof of Lebesgue's Theorem it is based on the following property¹ of continuous functions:

If the function $f(x)$ is continuous between a and b and in the points a and b of this interval, and b being included, take different values A and B , then for one or more determined values of x between a and b (b included) shall take any value C comprised between A and B .

For the purpose of the function x Dini's proof of the above property will be with the ordinary notion of continuity in which $f(x)$ is bounded, but with much smaller Professor Pompeiu's proof will not require this restriction, but only that $f(x)$ is differentiable.

For the purpose of the function x Dini's proof of $f(x)$ is continuous at $f(a)$ whereas for Professor Pompeiu's proof

$$R(x, a_1) = \frac{f(x) - f(a_1)}{x - a_1}$$

is taken to be $f(a)$ with the convention that $f(a) = f(a_1)$. It is to be open that in order that the above proof be valid, $f(x)$ must be continuous at the center a of $f(a)$ and the limit $f(a)$ must be a limit which is unattainable.

This question comes to be in conformity with the view held at the time by Professor Pompeiu himself. For in a paper² published after the publication of the paper containing the proof of Theorem 15, Professor Pompeiu says: "the proof which we have given rests on a property of continuous functions. Now this property has been extended in the present paper to generalised continuous functions and this extension may be applied to the general case. In fact let us consider the ratio

$$R(x, a) = \frac{F(x) - F(a)}{x - a}$$

and complete its definition by putting

$$R(a, a) = F'(a)$$

$f(x)$ may be extended at the point $x = a$, but in this case $f(a)$ is continuous in the extended sense at the point a .

¹ Dini's *Fondamenti per la teoria delle funzioni*, 1878, p. 1.

² Sur la continuité des courbes continues, *Revue des Nouvelles Académiques de l'Université de Jassy*, 1908.

APPENDIX B

THE VALUE OF THE REMAINDER IN LAGRANGE'S THEOREM

The theorem gives the remainder after a function f is expanded in a Taylor series (or is differentiated n times) in terms of $f^{(n+1)}$ evaluated at some value. The remainder is usually given by some writers in the form $R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$, where ξ is some value between a and x . In this Appendix first we shall find the value of the remainder and then we shall find the value of ξ in the case of the function $f(x) = e^x$.

List of the Various Forms

$$I. \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

$$[f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n, \quad 0 < \theta < 1]$$

$$II. \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

$$[f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n, \quad 0 < \theta < 1]$$

$$III. \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

$$[f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n, \quad 0 < \theta < 1]$$

$$IV. \quad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Let $f(x)$ be any continuous function with n derivatives on the interval $a \leq x \leq b$.

$$[f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n, \quad 0 < \theta < 1]$$

The above theorem is due to Lagrange. A more general theorem is given by Taylor. See also the book by L. E. Dickson, "The Theory of Numbers" (1919) pp. 41-42.

$$V \quad \{f(x) + f(x+h) + \dots + f(x+(n-1)h)\} = \frac{f(x) + f(x+nh)}{2} \cdot nh$$

$0 < \theta < 1$ is any number which satisfies the conditions of the differential coefficient of $f(x)$ at $x + \theta h$ when n is a natural number.

Roche's *Leçons de Calcul Différentiel*, 1881, p. 101.

$$VI \quad \frac{h^{n+1}}{n+1} \{f(x) + f(x+h)\} = \text{the quantity which tends to } 0 \text{ as } h \rightarrow 0$$

[Poisson, *Leçons de Calcul Différentiel*, 1842, p. 203.]

$$VII \quad \int_0^h f(x) dx = \int_0^h f(x+h) dx$$

[Lacroix, *Traité de Calcul Différentiel et Intégral*, 2nd edition, 1800, p. 309.]

$$VIII \quad \frac{h^{n+1}}{n+1} \int_0^1 f(x) dx = \frac{h^{n+1}}{n+1} \int_0^1 f(x+h) dx$$

[Lacroix, *Ibid.*, p. 309.]

$$IX \quad \frac{1}{n+1} \int_0^1 f(x) dx = \frac{1}{n+1} \int_0^1 f(x+h) dx$$

[Lacroix, *Traité de Calcul Différentiel et Intégral*, 1800, p. 309. Also *Goursat*, t. VII, p. 179.]

$$X \quad \frac{1}{n+1} \int_0^1 (h-t)^{n-1} f^{(n)}(x+t) dt$$

[Lacroix, *Traité de Calcul Différentiel et Intégral*, 1800, p. 309.]

The History of the Integral Calculus

The term of the integral calculus first came into use. It is found in *Les Leçons sur le calcul des fonctions* by Fourier, published in 1828 and published first in 1801 in the *Recueil des leçons de l'école normale* and again printed in 1804 in the *Journal de l'école polytechnique*. It may, however, be traced in the *Leçons de calcul différentiel et intégral* published in 1767 by the author, though the word *intégral* is not found in a definite article, the word *intégrer* being used in the sense of *équivalent*. It is also found in the *Leçons de calcul différentiel et intégral* by Lacroix, 1800, p. 309.

whence

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a).$$

Let us assume that the functions f and f' be continuous and remain finite and continuous in the interval from a to $a+h$ and that $\phi^{(n-1)}(a)$ is not zero, the same conditions as for the functions f and f' and one may apply to them the formula (1) which reduces now in this case to

$$\frac{f(a+h) - f(a) - hf'(a)}{h^n} = \frac{f^{(n)}(a+\theta h)}{n!}.$$

because

$$f(a+h) - f(a) - hf'(a) = \frac{h^n}{n!} \phi^{(n)}(a).$$

Hence follows at once

$$\frac{f(a+h) - f(a) - hf'(a)}{h^n} = \frac{\frac{h^n}{n!} \phi^{(n)}(a)}{\frac{h^n}{n!} \phi^{(n-1)}(a)} = \frac{\phi^{(n)}(a+\theta h)}{\phi^{(n-1)}(a+\theta h)}. \quad (2)$$

a relation of which the formula (1) is a particular case.

If now one puts

$$f(a+h) - f(a) - hf'(a) = \frac{h^n}{n!} \psi^{(n)}(a),$$

the equation (2) may be written

$$\psi^{(n)}(a) = \left\{ f(a+h) - f(a) - hf'(a) - \frac{h^n}{n!} \psi^{(n)}(a) \right\} \frac{f^{(n)}(a+\theta h)}{\phi^{(n-1)}(a+\theta h)}.$$

By giving to the arbitrary function ψ such form as we please (subject to the conditions enunciated above) we shall have at the expression for the remainder in Taylor's series. The equation (3) is therefore the new form of this remainder.

For example, if one puts

$$\psi(x) = x - a - h,$$

one finds

$$R = \frac{f^{(n+1)}(a+\theta h)}{(n+1)!} \frac{h^{n+1}}{\phi^{(n)}(a+\theta h)} = \frac{f^{(n+1)}(a+\theta h)}{(n+1)!} \frac{h^{n+1}}{\phi^{(n)}(a+\theta h)}.$$

where p and q are unetermined, $1 \leq p, q \leq \infty$, integral and h is any positive number. (1) can now be written very easily as an expression for the remainder.

In particular, for $p=q$,

$$R_n = \frac{h^{n+1}(1-\theta)^{n-p}}{n!(p+1)} f^{(n+1)}(\theta h),$$

a form in which I have given for representing f on a hotwired circuit. In fact, I represent x for $p=n$ and $p=\infty$ the ordinary remainder and the Lagrange form.

When one puts $p=\infty$ in the general expression (1) one obtains the formula of Schmidt:

$$f(x) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\theta h),$$

provided that in the interval $[a, a+h]$ $f^{(n+1)}(x)$ is not anywhere zero; in other words if in this interval $f^{(n+1)}(x)$ varies always from one to another without $f^{(n+1)}(x) = 0$ at $x = a + \theta h$ and then

$$1 \leq \theta \leq 1, \quad \theta \neq 0, \quad \{f^{(n+1)}(a + \theta h) \neq 0\}$$

which can be expressed thus: the remainder independent of n .

VI. In *Gesammelte Werke von Dr. Heinrich Poincaré*, 1901, 1902, 1903, 1904, 1905, 1906, 1907, 1908, 1909, 1910, 1911, 1912, 1913, 1914, 1915, 1916, 1917, 1918, 1919, 1920, 1921, 1922, 1923, 1924, 1925, 1926, 1927, 1928, 1929, 1930, 1931, 1932, 1933, 1934, 1935, 1936, 1937, 1938, 1939, 1940, 1941, 1942, 1943, 1944, 1945, 1946, 1947, 1948, 1949, 1950, 1951, 1952, 1953, 1954, 1955, 1956, 1957, 1958, 1959, 1960, 1961, 1962, 1963, 1964, 1965, 1966, 1967, 1968, 1969, 1970, 1971, 1972, 1973, 1974, 1975, 1976, 1977, 1978, 1979, 1980, 1981, 1982, 1983, 1984, 1985, 1986, 1987, 1988, 1989, 1990, 1991, 1992, 1993, 1994, 1995, 1996, 1997, 1998, 1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 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3719, 3720, 3721, 3722, 3723, 3724, 3725, 3726, 3727, 3728, 3729, 3730, 3731, 3732, 3733, 3734, 3735, 3736, 3737, 3738, 3739, 3740, 3741, 3742, 3743, 3744, 3745, 3746, 3747, 3748, 3749, 3750, 3751, 3752, 3753, 3754, 3755, 3756, 3757, 3758, 3759, 3760, 3761, 3762, 3763, 3764, 3765, 3766, 3767, 3768, 3769, 3770, 3771, 3772, 3773, 3774, 3775, 3776, 3777, 3778, 3779, 3780, 3781, 3782, 3783, 3784, 3785, 3786, 3787, 3788, 3789, 3790, 3791, 3792, 3793, 3794, 3795, 3796, 3797, 3798, 3799, 3800, 3801, 3802, 3803, 3804, 3805, 3806, 3807, 3808, 3809, 3810, 3811, 3812, 3813,

If the differential coefficient of $f(x)$ has the n th differential coefficient for $x = a$ then it ought to have the preceding differential coefficients also in the neighbourhood of $x = a$ but at least the n th differential coefficient it is not necessary to suppose either the existence or the continuity in the neighbourhood of $x = a$.

As stated on p. 70 of this book the term VI is also given in Dini's *Lezioni di Analisi*, 1 (1907) which practically contains the substance of Dini's lectures of middle or years. It is therefore unfortunate that recently two books¹ have come out in English in which the remainder VI is prominently described as Young's form of the remainder. No doubt this mistake is due to the ignorance of the author of the works of Peano and Dini and may be excused especially in view of the fact that so well informed a mathematician as Professor W. H. Young gives the remainder in his book *Elementary Theorems of the Differential Calculus* (1910) without mentioning Dini or Peano.

VII. There is also a gain in the discovery of this form that it affords the discovery that Dini's result is evidenced by the following remark². See how how Dini's result has obtained and demonstrated. At the same time the theory of Taylor (Recherches sur les fonctions continues et la systeme de monnaie 1910 p. 70).

That Dini's work is the work of Professor Pringhouse³ who says: A previously given derivation of Taylor's series by Dini's method shows that the publication of Euler's Differential Calculus is based on the (obviously inaccurate) relation⁴

$$f(x+h) = f(x) + \int_0^1 f'(x+kh)dh$$

and its repeated application to $f'(x+h)$, $f''(x+h)$, etc. With the repetition n times of the process of transformation would one attain to an expression for the remainder in Taylor's expression in the form of a repeated integral with n integrations of such a quality has Dini's result been explicitly and unambiguously.

- ¹ Phillips, *A Course of Analysis*, 1909;
- Mahajan, *Elementary Lessons in Analysis*, 1920.
- ² See Art. 284 of Lacroix's book
- ³ L.C., p. 459

- ⁴ The correct relation is $f(x+h) = f(x) + \int_0^1 f'(x+kh)dh$

VIII. (a) Although it is true that Lagrange was the first to give the remainder in the form of a definite integral, his form is fairly complicated and from it VIII can be derived only by a cumbersome process.

In fact Lagrange's statement giving his form of the remainder is included in the following ¹ :—

Let

$$f(x+h) = f(x) + f'(x) \cdot \frac{h}{1!} + \dots + f^{(n-1)}(x) \cdot \frac{h^{n-1}}{(n-1)!} + r_n(x, h) \cdot h^n,$$

then substitute $(x-h)$ for x and put

$$r_n(x-h, h) = q_n(x, h).$$

Thus

$$f(x) = f(x-h) + f'(x-h) \cdot \frac{h}{1!} + \dots + f^{(n-1)}(x-h) \cdot \frac{h^{n-1}}{(n-1)!} + q_n(x, h) \cdot h^n. \quad (1)$$

Put now

$$h = xs, \quad q_n(x, xs) = p_n(x, s),$$

then (1) gives

$$f(x) = f(x-xs) + f'(x-xs) \cdot \frac{xs}{1!} + \dots + f^{(n-1)}(x-xs) \cdot \frac{x^{n-1}s^{n-1}}{(n-1)!} + p_n(x, s) \cdot s^n. \quad (2)$$

Hence by partial differentiation with respect to s ,

$$0 = -f^{(n)}(x-xs) \cdot \frac{x^n s^{n-1}}{(n-1)!} + \frac{\partial p_n}{\partial s} \cdot s^n,$$

$$\text{i.e., } \frac{\partial p_n}{\partial s} = f^{(n)}(x-xs) \cdot \frac{x^{n-1}}{(n-1)!}.$$

Now, from (2), we have $p_n(x, 0) = 0$. Therefore, finally,

$$p_n(x, s) = \frac{1}{(n-1)!} \int_0^s y^{n-1} f^{(n)}(x-xy) dy. \quad (3)$$

If in (2) and (3) we write $\frac{s}{x}$ for s , we have

$$\begin{aligned} r_n\left(x, \frac{s}{x}\right) &= \frac{s^n}{(n-1)!} \int_0^1 y^{n-1} f^{(n)}(x-xy) dy \\ &= \frac{1}{(n-1)!} \int_0^1 t^{n-1} f^{(n)}(x-t) dt, \end{aligned}$$

which is practically the same as any of the forms VIII—X.

¹ This is taken almost word for word from Pringsheim's paper (l.c., pp. 441-442).

(b) Lacroix deduces VIII from VII by using the result

$$\int_0^1 H(h) dh^n = \frac{1}{(n-1)!} \int_0^1 (h-t)^{n-1} H(t) dt$$

(c) Laplace's derivation ¹ of his form is substantially reproduced by Cauchy in his *Résumé* (Leçon, 36) and is as follows:—

" One has by taking the integral from $x=0$,

$$\int dx \phi'(x-z) = \phi(x) - \phi(x-z),$$

$$\int dx \phi'(x-z) = z \phi'(x-z) + \int x dx \phi''(x-z),$$

$$\int x dx \phi''(x-z) = \frac{1}{2} z^2 \phi''(x-z) + \frac{1}{2} \int x^2 dx \phi'''(x-z).$$

By continuing this process, one finds generally

$$\begin{aligned} \int dx \phi'(x-z) &= z \phi'(x-z) + \frac{z^2}{1 \cdot 2} \phi''(x-z) + \dots + \frac{z^n}{n!} \phi^{(n+1)}(x-z) \\ &\quad + \int \frac{z^n dx}{n!} \phi^{(n+1)}(x-z). \end{aligned}$$

By comparing this expression with

$$\int dx \phi'(x-z) = \phi(x) - \phi(x-z),$$

one shall have

$$\phi(x) = \phi(x-z) + z \phi'(x-z) + \frac{z^2}{2!} \phi''(x-z) + \dots$$

Putting $x-z=t$, the preceding equation takes the form

$$\begin{aligned} \phi(t+z) &= \phi(t) + z \phi'(t) + \frac{z^2}{2!} \phi''(t) + \dots + \frac{z^n}{n!} \phi^{(n+1)}(t) \\ &\quad + \frac{1}{n!} \int_t^{t+z} x^n dx \phi^{(n+1)}(t+x-z). \end{aligned}$$

APPENDIX C.

ADDITIONS AND CORRECTIONS.

Page 4.	Line 17:	For	" $f(x_0 + h) f(x_0)$ "	read	" $f(x_0 + h) - f(x_0).$ "
Page 5.	2nd foot-note:	For	" 78 "	read	" 75."
Page 9.	Line 12:	For	"also nowhere"	read	"also generally nowhere."
Page 11.	Line 2:	For	"Pompien"	read	"Pompeio."
Page 11.	1st foot-note:	For	"theorème"	read	"théorème."
Page 19.	Line 18:	For	" $(x\phi +$ "	read	" $\phi'(x +$ "
Page 19.	1st foot-note:	For	" $x \triangle x$."	read	" $x \triangle x$."
Page 22.	Line 6 from bottom:	For	" $F(\xi)$ "	read	" $F(\xi)$."
Page 23.	3rd line from bottom:	For	" $\frac{\phi(x' - \xi) - \phi(x')}{\xi}$ "	read	" $\frac{\phi(x' - \xi) - \phi(x')}{-\xi}$."
Page 26.	at the end: Add as a foot-note to line 6:				
<p>"Although, as shown in Appendix A, the statement in this sentence is correct, the example that follows is not to the point. Professor Pompeio was good enough to respond to the author's request to express his opinion on the criticism and draw the author's attention to the second paper quoted on p. 97 in the 2nd foot-note. The proof, as modified in the light of this paper, is perfectly valid."</p>					
Page 30.	Line 20:	For	" $f(\xi)$ is"	read	" $f(\xi)$ is."
Page 30.	foot-note:	For	"Ibid"	read	"See the foot-note on the preceding page."
Page 31.	Line 18:	For	" $f(b - 0)$ "	read	" $f(b - 0)$."
Page 33.	Line 8 from bottom:	For	" ϕx "	read	" $\phi(x)$."
Page 44.	Line 1:	For	"immediately"	read	"immediately."
Page 49.	Line 9:	For	" $2A_0A_1$ "	read	" $2A_0A_2$."
Page 51.	Line 3:	For	" $\{f^{(2)}\}^2 f^{(3)}$ "	read	" $\{f^{(2)}\}^2, f^{(3)}$."
Page 51.	Line 9:	For	" $f^{(2)}$ "	read	" $f^{(3)}$."
Page 53.	Line 2:	For	"left"	read	"right."
Page 53.	Line 11:	For	"(2)"	read	"(3)."
Page 53.	Line 16:	For	"equation and"	read	"equation by $f''(x)$ and."
Page 54.	Line 3:	For	"left"	read	"right."

- Page 54. last line: For " $h'x + \xi$ " read " $+ h'(x + \xi)$."
- Page 60. 7th line from bottom: For $\frac{\cos \psi \left(\frac{h}{2} \right)}{h^2 \psi \left(\frac{h}{2} \right)}$ read $\frac{\cos^2 \psi \left(\frac{h}{2} \right)}{2h^2 \psi \left(\frac{h}{2} \right)}$.
- Page 62. Add as foot-note to ξ_m in the last line: " ξ_m is the same as ω_m ."
- Page 64. Line 10: For "difficult" read "not difficult."
- Page 67. Line 10: For " $x = \frac{2}{(4n+1)\pi}$ " read " $x = \frac{2}{(4n+1)\pi}$ "
- Page 68. Line 7: For "the maxima" read "the proper maxima."
- Page 68. 2nd line from bottom: For "33" read "32."
- Page 74. Line 2: For "32" read "31."
- Page 75. Foot-note: For "second paper" read "second and third papers."
- Page 76. At the end of Art. 59 add:

The subject is carefully considered in Prasad's third paper (*Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, pp. 57-66). No doubt, $\frac{d\xi}{dh}$ cannot be finite and different from zero or infinite with determinate sign. But there is some doubt if $\frac{d\xi}{dh}$ can be zero at a point h for which $f(h)$ has a cusp; Prasad's opinion is that most probably $\frac{d\xi}{dh} = 0$ at such a point.

- Page 80. Line 1: For "values" read "of values."
- Page 80. Line 5 from bottom: For " $a_0 \xi$ " read " a_0 ."
- Page 83. Line 15: For " $\cos^2 \beta x$ " read " $\cos^2 \beta x + \beta x$."
- Page 85. Line 3 from bottom: For " h " read " h ."
- Page 87. Line 4: For $\frac{\partial \xi}{\partial x}$ read $\frac{\partial \xi}{\partial x_1}$.
- Page 87. Line 3 from bottom: For $(x_2 - x_1) f'(\xi)$ read $(x_2 - x_1) f''(\xi)$.
- Page 87. Line 3 from bottom: For $f'(\xi_1 - \xi_2)$ read $f'(\xi_1 - \xi_2)$.
- Page 89. Line 11 from bottom: For " $(Fx + h)$ " read " $F(x + h)$."
- Page 90. Line 5: For " h^n " read " h^n ."